

RENEWAL SEQUENCES, DISORDERED POTENTIALS, AND PINNING PHENOMENA

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ABSTRACT. We give an overview of the state of the art of the analysis of disordered models of pinning on a defect line. This class of models includes a number of well known and much studied systems (like polymer pinning on a defect line, wetting of interfaces on a disordered substrate and the Poland-Scheraga model of DNA denaturation). A remarkable aspect is that, in absence of disorder, all the models in this class are exactly solvable and they display a localization-delocalization transition that one understands in full detail. Moreover the behavior of such systems near criticality is controlled by a parameter and one observes, by tuning the parameter, the *full spectrum* of critical behaviors, ranging from first order to infinite order transitions. This is therefore an ideal set-up in which to address the question of the effect of disorder on the phase transition, notably on critical properties. We will review recent results that show that the physical prediction that goes under the name of *Harris criterion* is indeed fully correct for pinning models. Beyond summarizing the results, we will sketch most of the arguments of proof.

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1. PINNING AND DISORDER: MODELS AND MOTIVATIONS

1.1. The basic example: pinning of simple random walks. It is somewhat customary to introduce pinning models by talking of pinning of simple random walks (SRW). This is due to a number of reasons, like the widespread grasp on SRW, or the fact that modeling several pinning phenomena naturally leads to random walk pinning, as we will see. However, we will see also that, in a sense, the SRW case is the hardest to deal with: nonetheless, we are going to follow the tradition and start from SRW pinning.

Let $S := \{S_n\}_{n=0,1,\dots}$ be a sequence of random variables such that $S_0 = 0$ and such that $\{S_n - S_{n-1}\}_{n=1,2,\dots}$ are IID (i.e., independent and identically distributed) symmetric random variables taking only the values $+1$ and -1 . The *disorder* is given by a sequence $\omega := \{\omega_n\}_{n=1,2,\dots}$ of real numbers and we will play with two real parameters β and h . We actually assume that ω is a realization of an IID sequence of standard Gaussian variables (see Remark 1.5 for some comments on generalizations). We call \mathbb{P} the law of ω and we denote by θ the left-shift operator on $\mathbb{R}^{\mathbb{N}}$: $(\theta\omega)_n = \omega_{n+1}$. Our aim is to study the probability measure $P_{N,\omega}$ (N is a positive integer: we will be interested in the limit $N \rightarrow \infty$) defined as

$$P_{N,\omega}(s_0, s_1, \dots, s_N) := \frac{1}{Z_{N,\omega}} \exp \left(\sum_{n=1}^N (\beta\omega_n + h) \mathbf{1}_{s_n=0} \right) P(s_0, s_1, \dots, s_N), \quad (1.1)$$

where

- (1) $P(s_1, s_2, \dots, s_N) = (1/2)^N$ if and only if $s_0 = 0$ and $|s_n - s_{n-1}| = 1$ for $n = 1, 2, \dots, N$;
- (2) $Z_{N,\omega}$ is the normalization constant (*partition function*), that is

$$Z_{N,\omega} = \sum_{s_0, s_1, \dots, s_N} \exp \left(\sum_{n=1}^N (\beta\omega_n + h) \mathbf{1}_{s_n=0} \right) P(s_0, s_1, \dots, s_N). \quad (1.2)$$

We will actually prefer a slightly different definition of the model, namely given the sequence $s = \{s_0, s_1, \dots\}$ we set

$$\frac{d\mathbf{P}_{N,\omega}}{d\mathbf{P}}(s) := \frac{1}{Z_{N,\omega}} \exp \left(\sum_{n=1}^N (\beta\omega_n + h) \mathbf{1}_{s_n=0} \right). \quad (1.3)$$

Notice that this time $\mathbf{P}_{N,\omega}$ is a measure on (infinite) sequences, namely the trajectory of the walk all the way to infinity, while in (1.1) we had defined a measure only up to *step* (or *time*) N . As a matter of fact, if we consider *cylindrical* events of the type $E = \{s = \{s_n\}_{n=0,1,\dots} : s_0 = t_0, s_1 = t_1, \dots, s_N = t_N\}$, then the measure of E under $\mathbf{P}_{N,\omega}$ defined in (1.3) coincides with $P_{N,\omega}(t_0, t_1, \dots, t_N)$.

Remark 1.1. It is worth stressing that, unless $\beta = 0$, in this model there are two sources of randomness: the polymer chain is modeled by a random walk with law \mathbf{P} and the disorder is a typical realization of the random sequence ω with law \mathbb{P} . These two sources of randomness are treated in very different ways: ω is *quenched*, that is chosen once and for all, while the polymer location fluctuates and in fact we study the distribution of S under $\mathbf{P}_{N,\omega}$.

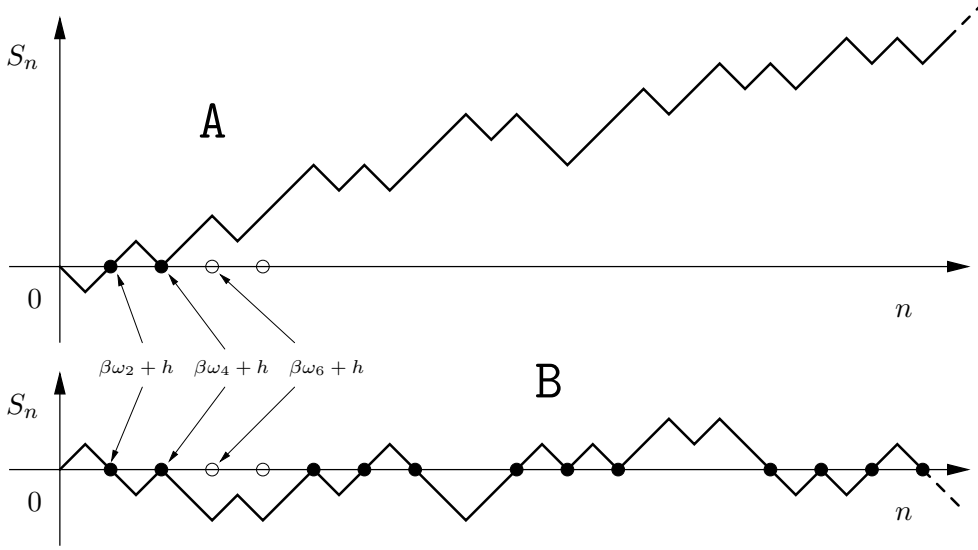


FIGURE 1. Two trajectories sampled from $\mathbf{P}_{N,\omega}$ for different values of β and h , with N very large, and represented as directed polymers, in the sense that we plot the function $n \mapsto S_n$ or, equivalently, we look at the walk $\{(n, S_n)\}_{n=0,1,\dots}$ with one deterministic component. The case A sketches a delocalized trajectory that is what one observes when h is, for example, negative and large (one should first think of the homogeneous case $\beta = 0$). Since the SRW is a periodic Markov chain, the origin is visited only at even times, so only $\omega_2, \omega_4, \omega_6, \dots$ play a role and they are the only *charges* (a charge at n is the quantity $\beta\omega_n + h$) marked in the drawing, with a filled circle if they are visited and with an empty circle if they are not visited: not all the unvisited charges are marked). The distinctive aspect of case A is that there are just a few *contacts*, i.e. visits to the origin, and then the walk resembles a walk conditioned not to hit zero. The case B is instead what one observes when h is positive and large: the number of contacts is large, as a matter of fact the drawing wants to suggest that there is a positive density of contacts. It is natural to call such a regime *localized*, in contrast to the previous one that we call *delocalized*. Note that both case A and B are atypical for the free walk ($\beta = h = 0$) in which there is a zero density of contacts but they are spread through the system and the walk certainly does not stay on one side of the axis as in case A. It is important to remark that we have a full up-down symmetry and this implies that in the delocalized regime A the walk is either delocalized above or below the axis with probability 1/2. Of course at this stage it is highly unclear that one observes either localized or delocalized trajectories (for typical ω) and to a certain extent this is not correct because one has to exclude the so called *critical regime*, which however appears only at exceptional values of β and h (the *phase transition* point, or *critical point*). We have of course avoided the delicate issue of the role of disorder, at the hearth of this presentation. Here we will simply content ourselves with pointing out that, for example, even when h is very large and negative (pushing thus toward delocalization) any amount of disorder, i.e. $\beta > 0$, yields a positive density of sites in which $\beta\omega_n + h$ is positive and therefore attractive. There could therefore be a *smart targeting strategy* of the polymer in placing the contacts at these sites, leading thus to localization.

As it is well known, the Markov process S is null-recurrent, namely every site of the state space \mathbb{Z} is visited (infinitely often) \mathbf{P} -almost surely, but the expectation of the time between successive visits to a given site is infinite. Let us be more explicit about this last concept and let us introduce, for $m \in \mathbb{Z}$, the random variable $\tau_1(m) := \inf\{n > 0 : S_n = m\}$ and, for $j > 1$, also $\tau_j(m) := \inf\{n > \tau_{j-1}(m) : S_n = m\}$. It is then a direct consequence of

the (strong) Markov property that $\{\tau_{j+1}(m) - \tau_j(m)\}_{j=1,2,\dots}$ is a sequence of IID random variables. It should be also clear that the law of $\{\tau_{j+1}(m) - \tau_j(m)\}_{j=1,2,\dots}$ does not depend on the value of m : we are going to denote $\tau_j(0)$ simply by τ_j and, since $S_0 = 0$, we are setting $\tau_0 = 0$. The recurrent character of S boils simply down to the fact that $\sum_n \mathbf{P}(\tau_1 = n) = 1$ and the recurrence is of null type because $\mathbf{E}[\tau_1] = +\infty$: the distribution of τ_1 is known in detail [23, Ch. III] and in particular

$$\mathbf{P}(\tau_1 = 2n) \stackrel{n \rightarrow \infty}{\sim} \frac{c}{n^{3/2}}, \quad (1.4)$$

where $c = 1/\sqrt{4\pi}$ and we have introduced the notation $a_n \stackrel{n \rightarrow \infty}{\sim} b_n$ for $\lim_{n \rightarrow \infty} a_n/b_n = 1$. Since (clearly) $\mathbf{E}[\tau_1] = \infty$, the classical Renewal Theorem (see (1.11) below) tells us that the expected number of visits to 0 of S up to time N is $o(N)$ (a more precise analysis shows that it is of the order of \sqrt{N} , see [23, Ch. III] or Theorem 1.2 below).

As we shall see, the trajectories of the process S are very strongly affected if β or h are non zero and, except for *critical cases*, what happens is roughly that, in the limit as $N \rightarrow \infty$, under $\mathbf{P}_{N,\omega}$ the expected number of the visits paid by S to 0 is of the order of N , or it is much smaller than \sqrt{N} (in some cases one can show that they are $O(1)$). A first glimpse at these different scenarios can be found in Figure 1.

1.2. The general model: renewal pinning. We have introduced the τ sequence in the previous subsection in order to give some intuition about the model, but its interest goes well beyond. A look at (1.3) suffices to realize that $Z_{N,\omega}$ can be expressed simply in terms of τ :

$$Z_{N,\omega} = \mathbf{E} \exp \left(\sum_{n=1}^N (\beta \omega_n + h) \mathbf{1}_{n \in \tau} \right), \quad (1.5)$$

where we have introduced a notation that comes from looking at $\tau = \{\tau_j\}_{j=0,1,\dots}$ as a random subset of $\mathbb{N} \cup \{0\}$, so that $n \in \tau$ means that there exists j such that $\tau_j = n$. Therefore the model in (1.3) is just a particular case of when τ is a general discrete renewal:

$$\frac{d\mathbf{P}_{N,\omega}^{\mathbf{f}}}{d\mathbf{P}}(\tau) := \frac{1}{Z_{N,\omega}^{\mathbf{f}}} \exp \left(\sum_{n=1}^N (\beta \omega_n + h) \mathbf{1}_{n \in \tau} \right), \quad (1.6)$$

where the superscript \mathbf{f} , that stands for *free*, has been introduced because a slightly different version of the model is going to be relevant too:

$$\frac{d\mathbf{P}_{N,\omega}^{\mathbf{c}}}{d\mathbf{P}}(\tau) := \frac{1}{Z_{N,\omega}^{\mathbf{c}}} \exp \left(\sum_{n=1}^N (\beta \omega_n + h) \mathbf{1}_{n \in \tau} \right) \mathbf{1}_{N \in \tau}, \quad (1.7)$$

and \mathbf{c} stands for *constrained*. Let us stress that by (discrete) renewal process τ we simply mean a sequence of random variables with (positive and integer valued) IID increments: we call these increments *inter-arrival* variables.

We will not be interested in the most general discrete renewal, but we will rather focus on the case in which

$$K(n) := \mathbf{P}(\tau_1 = n) = \frac{L(n)}{n^{1+\alpha}}, \quad n \in \mathbb{N} = \{1, 2, \dots\} \quad (1.8)$$

where α is a positive number and

$$\lim_{n \rightarrow \infty} L(n) = c_K > 0. \quad (1.9)$$

We call $K(\cdot)$ *inter-arrival distribution*. Note that we always assume $K(0) = 0$. Moreover we will assume that $\sum_{n \in \mathbb{N}} K(n) \leq 1$: the case $\sum_{n \in \mathbb{N}} K(n) < 1$ has to be interpreted as the case of a *terminating* renewal, in the sense that $K(\infty) := 1 - \sum_{n \in \mathbb{N}} K(n)$ is the probability that $\tau_1 = +\infty$. Therefore, if $K(\infty) > 0$, the cardinality $|\tau|$ of the random set τ is almost surely finite. The case of $K(\infty) = 0$ is instead the case of a *persistent* renewal, and $|\tau| = \infty$ almost surely. But persistent renewals are of two different kinds: they are *positive* persistent if $\sum_n nK(n) (= \mathbf{E}\tau_1) < \infty$, or *null* persistent if the same quantity diverges. This terminology reflects the fact that for any *aperiodic* renewal (aperiodicity refers to the fact that τ_1 does not concentrate on a sublattice of \mathbb{N}) the law of large numbers ensures that almost surely

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\tau \cap [0, n]| = \frac{1}{\mathbf{E}\tau_1} \in [0, 1], \quad (1.10)$$

so that if $\mathbf{E}\tau_1 = \infty$ we are facing a *zero density* renewal. We stress that the last statement holds also for terminating renewals for which $\mathbf{E}[\tau_1] = \sum_{n \in \mathbb{N}} nK(n) + \infty K(\infty) = \infty$ (note on the way that, for us, $\sum_n \dots$ never includes $n = \infty$).

Very relevant for the analysis of renewal processes is the renewal function $n \mapsto \mathbf{P}(n \in \tau)$, that is the probability that the site n is visited by the renewal. We call such a function $K(\cdot)$ -*renewal function* when we want to be more precise. The asymptotic behavior of the renewal function is captured by the so called Renewal Theorem (for a proof see e.g. [8]). This theorem says that if τ is an aperiodic renewal (the generalization to the periodic case is immediate) we have

$$\lim_{n \rightarrow \infty} \mathbf{P}(n \in \tau) = \frac{1}{\mathbf{E}\tau_1} \in [0, 1]. \quad (1.11)$$

Note the link with (1.10), but note also that this is little informative if $\mathbf{E}\tau_1 = \infty$. The leading asymptotic behavior in such a case is summed up in the following statement that calls for the definition of the Gamma function: $\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) dt$, for $x > 0$. Recall that in our set-up $\mathbf{E}[\tau_1] = \infty$ either because $K(\infty) > 0$ (terminating renewal), regardless of the value of α , or because $\alpha \leq 1$.

Proposition 1.2. *Assuming (1.8) and (1.9) we have:*

(1) *if $K(\infty) > 0$ then*

$$\mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \frac{K(n)}{K(\infty)^2}. \quad (1.12)$$

(2) *If $K(\infty) = 0$ and $\alpha \in (0, 1)$ then*

$$\mathbf{P}(n \in \tau) \stackrel{n \rightarrow \infty}{\sim} \left(\frac{\alpha \sin(\pi\alpha)}{\pi C_K} \right) n^{\alpha-1}. \quad (1.13)$$

Proposition 1.2(1) is a classical result detailed for example in [9] or [29, § A.6]. Proposition 1.2(2) is instead more delicate and while the case $\alpha \in (1/2, 1)$ is under control since [27], the full case is instead a rather recent result [21]. Note however that if full proofs are non-trivial, one can rather easily find intuitive arguments suggesting the validity of Proposition 1.2 [29, § A.6].

The asymptotic behavior for the case $\alpha = 1$ is known too, but this case is a bit anomalous and, in this review, we will often skip the results for $\alpha = 1$ that would make the exposition heavier.

Remark 1.3. In full generality, given a renewal τ one can find a Markov process S on a state space containing a point 0 such that $\tau_1 = \inf\{n \in \mathbb{N} : S_n = 0\}$. Still in full generality, the state space can be chosen equal to $\mathbb{N} \cup \{0\}$, see [29, App. A.5] for details. Therefore, with this remark in mind, one could go back to the original definition (1.3) without loss of generality.

Remark 1.4. Everything we are going to present works assuming simply that $K(\cdot)$ is *regularly varying* or, equivalently, that $L(\cdot)$ is *slowly varying*. Examples of slowly varying functions include $\log(n)^c$, c a real number, or any product of powers of iterated logarithms (see [9] for full definitions and properties or [29, § A.4] for a quick sum-up). Regularly varying functions are a natural set-up for pinning models also because some natural cases do involve slowly varying functions: for example the law of τ_1 for the two dimensional simple random walk one has $K(n) \stackrel{n \rightarrow \infty}{\sim} c/(n(\log n)^2)$, for an explicit value of $c > 0$ [40].

In this subsection we have focused on the behavior of the free system: $\beta = h = 0$. The observations made in the caption of Figure 1 do apply to the general case too (that is to $(\beta, h) \neq (0, 0)$), with, nevertheless, two distinctions:

- (1) if $\alpha > 1$ and if the renewal is persistent, the free renewal is already localized, since by the Renewal Theorem the contact points have a positive density. We will see that this affects the discussion in the caption of Figure 1 only for what concerns the critical case, but the general picture still holds;
- (2) if τ is terminating it is of course harder to localize the process, but in reality it is rather easy to show that the model can be mapped to a persistent τ case, precisely if one sets $\tilde{K}(n) = K(n)/(1 - K(\infty))$ and if $\tilde{\tau}$ is the (persistent!) $\tilde{K}(\cdot)$ -renewal

$$Z_{N,\omega}^c = \mathbf{E} \left[\exp \left(\sum_{n=1}^N (\beta \omega_n + h + \log(1 - K(\infty))) \mathbf{1}_{n \in \tilde{\tau}} \right); N \in \tilde{\tau} \right], \quad (1.14)$$

and the same is true also at the level of the measure $\mathbf{P}_{N,\omega}$ itself. The proof of such a statement is absolutely elementary and it is detailed for example in [29, Ch. 1]: note that, from a mathematical standpoint, this allows us to restrict ourselves to persistent renewals τ .

1.3. A gallery of applications. The localization mechanism captured by class of models we have just introduced comes up in modeling a variety of phenomena. Here we just extract some examples and cite some references.

Polymers and defect lines. The interaction between polymers, chains of elementary units called monomers, and the surrounding medium or other polymers is omnipresent in physics, chemistry and biology. We cite for example [28], but it is of course impossible to account for the literature in such a direction. The case we are interested in is the one in which a polymer is fluctuating in a neutral medium except for a line, or a tube, with which the polymer interacts. Actually, also cases in which the line is for example a surface or even simply a point may be modeled by the type of pinning models we are considering. The key point is that polymers are often modeled by self-avoiding random walks and a simplified way to impose the self-avoiding condition is considering directed walks (see Figure 1). So the polymer pinning model becomes precisely the renewal pinning we are considering (we refer to [17, 24, 26, 42, 57] for examples of the phenomena that are modeled via directed

walk pinning). Here we just stress that the dimensionality of the problem enters the renewal pinning only via the exponent α : for example a polymer in three dimensions pinned to a line can be modeled by $\{(n, S_n)\}_{n=0,1,\dots}$, where S is a random walk in two dimensions, for which $\alpha = 0$ (see Remark 1.4). The general case of a physical space of $d+1$ dimensions leads to $\alpha = (d/2) - 1$, for $d \geq 2$, and of course $\alpha = 1/2$ if $d = 1$.

Wetting phenomena. Modeling interfaces in two dimensional media by random walks has a long history [1] that is somewhat summed up also in [29, App. C] in which one can find the explanation of why anisotropic Ising models do reduce in a suitable limit to the renewal pinning model with $\alpha = 1/2$. A particular choice of the boundary conditions leads to the so called wetting problem [13, 46], which is just the case in which the random walk trajectories that one considers are only the ones above (and touching) the axis: with reference to Figure 1, to obtain allowed trajectories one has to flip over the negative excursions. As it is explained in detail in [29, Ch. 1], this problem just corresponds to renewal pinning with $\alpha = 1/2$ and $K(\infty) = 1/2$ and, at the level of contact points, the process can be mapped to the case in Figure 1 with h replaced by $h - \log 2$ (see (1.14)).

DNA denaturation: the Poland-Scheraga model. Two-stranded DNA has been often modeled by two directed walks with pinning potentials, see e.g. [47] and references therein. Since the difference of two independent random walks is still a random walk, we are back to renewal pinning. However directed walk models lead to values of α that are in contrast with observation, so that a considerable amount of work has been put into understanding whether (in our language) renewal pinning is a reasonable model and which α should be chosen (see in particular [41], but once again we refer to [29] for a more complete bibliography). We note that inhomogeneous or disordered modeling is really required in this context, because the pinning strength does depend on the type of base pair, see Figure 2. The renewal pinning model with inhomogeneous charges has been and is extensively used for the study of DNA denaturation [11, 17]: appropriate values of α are close to 1.15.

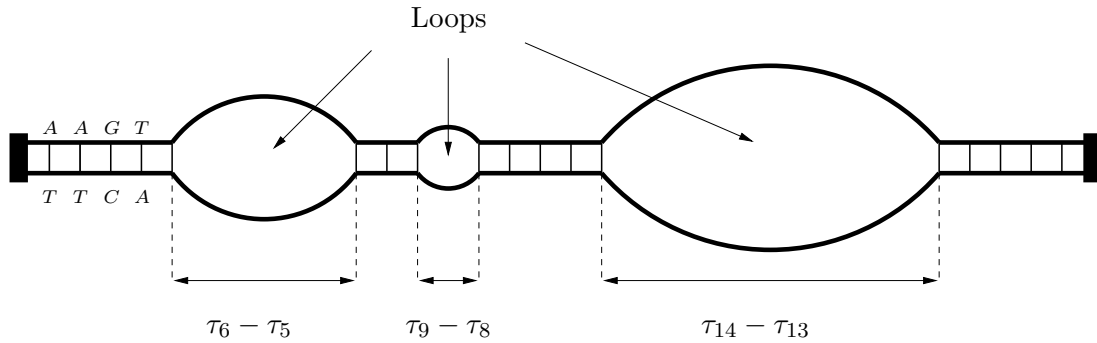


FIGURE 2. The two thick lines are the DNA strands. They may be paired, gaining thus energetic contributions that depend on whether the base pair is A-T or G-C (the model is therefore inhomogeneous). There are then sections of unpaired bases (the *loops*) to which an entropy is associated. The DNA portion in the drawing corresponds to the renewal model trajectory with $\tau_j - \tau_{j-1} = 1$ except three τ -interarrivals (so the loops correspond to inter-arrival of length 2 or more).

Remark 1.5. For DNA denaturation taking ω_1 Gaussian is not appropriate. In this case ω_1 should rather be a binary variable. To be more precise one should also take into account

stacking energies, that is energies depend on blocks of two pairs, and probably one should also study correlated sequences of bases. Sticking to the issue of binary variables versus Gaussian ones, we take this occasion to stress that much of the mathematical literature is written for rather general charge distribution (say, with finite exponential moments of all orders).

2. THE HOMOGENEOUS CASE

The full solution of the non disordered ($\beta = 0$), or *homogeneous*, case is crucial and, at the same time, it is rather elementary once it is phrased in the renewal theory language. Such a solution has been repeatedly presented in the physical literature in several particular instances, by using what a probabilist would call *generating function techniques*: in particular one can find a very nice and complete presentation in [24]. However the presentation we are going to outline in detail here is different and much more direct.

In this section, but also later, with abuse of notation we will denote by $Z_{N,h}^a$ ($a = \mathbf{c}, \mathbf{f}$) the partition function $Z_{N,\omega}^a$ when $\beta = 0$. Let us start by observing that we can write

$$Z_{N,h}^{\mathbf{c}} = \sum_{n=1}^N \sum_{\substack{\ell \in \mathbb{N}^n: \\ \sum_{j=1}^n \ell_j = N}} \prod_{j=1}^n \exp(h) K(\ell_j). \quad (2.1)$$

Note that if $h = 0$ then $Z_{N,h}^{\mathbf{c}} = \mathbf{P}(N \in \tau)$, i.e. the partition function is just the $K(\cdot)$ -renewal function. The right-hand side of (2.1) is still a renewal function if $e^h K(\cdot)$ is an inter-arrival law. And indeed it is if $\sum_{n \in \mathbb{N}} e^h K(n) \leq 1$ and in this case $Z_{N,h}^{\mathbf{c}}$ is the $e^h K(\cdot)$ -renewal function: its asymptotic behavior is hence given in Theorem 1.2, but we prefer to delay such a result since a unified approach holds for every h . In fact if $\sum_{n \in \mathbb{N}} e^h K(n) \geq 1$ we can renormalize the expression by introducing an exponential correction, going back to a renewal function (times an exponentially growing factor). Precisely we call $b(\geq 0)$ the (unique) solution of

$$\sum_{n \in \mathbb{N}} \exp(-bn + h) K(n) = 1, \quad (2.2)$$

and we set $K_b(n) := \exp(-bn + h) K(n)$. We have therefore defined a function $h \mapsto b(h)$ for h such that $\sum_{n \in \mathbb{N}} e^h K(n) \geq 1$, that is for $h \geq h_c(0)$, with

$$h_c(0) := -\log \sum_n K(n). \quad (2.3)$$

For $h < h_c(0)$ we set instead $b(h) = 0$ and $K_0(n) := \exp(h) K(n)$ (of course the latter notation is poor since h is not explicit). With this notations we can write

$$Z_{N,h}^{\mathbf{c}} = \exp(bN) \sum_{n=1}^N \sum_{\substack{\ell \in \mathbb{N}^n: \\ \sum_{j=1}^n \ell_j = N}} \prod_{j=1}^n K_b(\ell_j) = \exp(bN) \mathbf{P}_h(N \in \tau), \quad (2.4)$$

where, under \mathbf{P}_h , τ is a $K_{b(h)}(\cdot)$ -renewal. By the Renewal Theorem

$$\lim_{N \rightarrow \infty} \mathbf{P}_h(N \in \tau) = \frac{1}{\mathbf{E}_h[\tau_1]}, \quad (2.5)$$

which is a positive constant if $h > h_c(0)$, but it is zero if $h < h_c(0)$ because the $K_{b(h)}(\cdot)$ -renewal is terminating. For $h = h_c(0)$ this limit may or may not be zero, but let us postpone this issue to Remark 2.1. Let us focus for now on the fact that for $h < h_c(0)$ the

Renewal Theorem does not yield the leading behavior, but thanks to Proposition 1.2(1) we see that

$$\mathbf{P}_h(N \in \tau) \stackrel{N \rightarrow \infty}{\sim} \frac{K(N)}{(1 - \exp(h)(1 - K(\infty)))^2}, \quad (2.6)$$

With these results in our hands we see

- (1) that since $N^{-1} \log \mathbf{P}_h(N \in \tau)$ vanishes as $N \rightarrow \infty$, we have therefore proven that

$$F(0, h) := \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N,h}^c = b(h), \quad (2.7)$$

for every h . The quantity $F(0, h)$ is usually called *free energy (per unit volume)* and of course we use such a notation because later there will be $F(\beta, h)$;

- (2) that (2.4) goes well beyond Laplace asymptotics: this is very relevant and allows us for example to compute the limit of

$$\mathbf{P}_{N,h}^c(\tau_1 = n_1, \tau_2 = n_1 + n_2, \dots, \tau_j = n_1 + \dots + n_j) = \prod_{i=1}^j (e^h K(n_i)) \frac{Z_{N-n_1-\dots-n_j,h}^c}{Z_{N,h}^c}, \quad (2.8)$$

as $N \rightarrow \infty$. For example when $h > h_c(0)$ the ratio of partition functions in the right-hand side converges to $\exp(-(n_1 + \dots + n_j)F(0, h))$ and therefore the all expression converges to $\prod_{i=1}^j K_{F(0,h)}(n_i)$. It is rather easy to see that the same holds also for $h < h_c(0)$.

Remark 2.1. In the above list we have been a bit clumsy about the critical case $h = h_c(0)$, but in reality what happens at $h = h_c(0)$ is clear too. First of all, $\sum_n K_{F(0,h_c(0))}(n) = 1$, so that the associated renewal is persistent. More precisely $K_{F(0,h_c(0))}(\cdot) = K(\cdot)$ if $\sum_n K(n) = 1$, and $Z_{N,\omega}^c = \mathbf{P}(N \in \tau)$, and otherwise $K_{F(0,h_c(0))}(\cdot)$ is just a multiple of $K(\cdot)$ and $Z_{N,\omega}^c$ coincides with the $K_{F(0,h_c(0))}(\cdot)$ -renewal function computed in N . Recall now that the $K_{F(0,h_c(0))}(\cdot)$ -renewal function converges to a positive constant if $\alpha > 1$ and to zero otherwise. But when it converges to zero there is Proposition 1.2(2) that comes to our help so that once again we know the sharp asymptotic behavior of $Z_{N,\omega}^c$. In particular $F(0, h_c(0)) = 0$.

All these remarks are telling us in particular that (2.2) is a formula for the free energy, in the sense that $F(0, h) = b$ if there exists a positive solution b to (2.2) and otherwise $F(0, h) = 0$. From such a formula one can extract a number of consequences that are summed up in the caption of Figure 3. In particular the behavior of the free energy near criticality is trivial for $h < h_c(0)$, but it is not for $h > h_c(0)$: let us make explicit the behavior of $F(0, h)$ as $h \searrow h_c(0)$. If $\sum_n nK(n) < \infty$ and if $h_c(0) = 0$ (which we may assume without loss of generality: recall (1.14)!))

$$\sum_n (1 - \exp(-b(h)n))K(n) = 1 - \exp(-h) \stackrel{h \searrow 0}{\sim} h. \quad (2.9)$$

The asymptotic behavior of the left-hand side is easily obtained since for every fixed n the limit of $(1 - \exp(-b(h)n))/b$ as $b \searrow 0$ is n . On the other hand $1 - \exp(-x) \leq x$ for every $x \geq 0$, so that the Dominated Convergence Theorem yields that the left-most side in (2.9) is asymptotically equivalent to $b \sum_n nK(n)$, and therefore $b(h) \sim h / \sum_n nK(n)$.

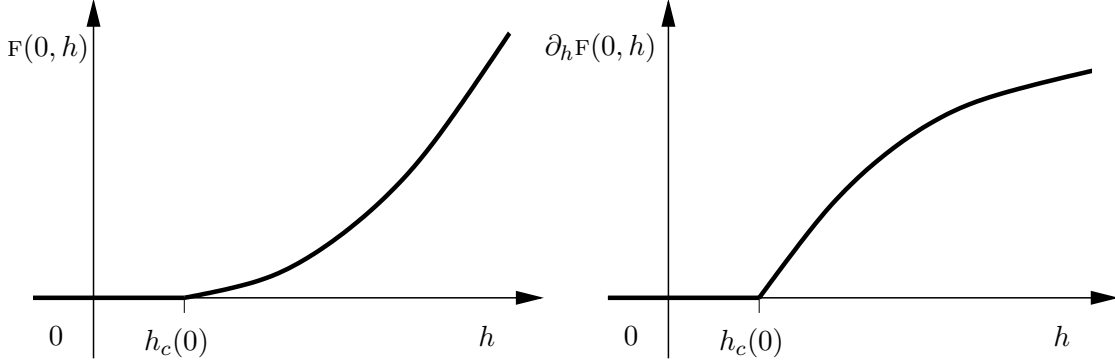


FIGURE 3. The function $h \mapsto F(0, h)$ is non decreasing, convex and non negative (convexity follows either from (2.2) or (2.7)). It is therefore equal to 0 up to $h_c(0) = \sup\{h : F(0, h) = 0\}$ and after this point it is positive and strictly increasing. Of course $h = h_c(0)$ is a point of non analyticity of the map $h \mapsto F(0, h)$: this map is of course analytic on $(-\infty, h_c(0))$. It is analytic also on $(h_c(0), \infty)$, by the Implicit Function Theorem for analytic functions. The graph of $\partial_h F(0, h)$ indicates that we are considering the case $\alpha = 1/2$: $F(0, \cdot)$ is C^1 but not C^2 . For $\alpha \in (1/2, 1]$ the slope of $h \mapsto \partial_h F(0, h)$ at $h_c(0)$ is infinite and for $\alpha > 1$ a jump discontinuity appears. We stress that $\partial_h F(0, h)$ is the contact fraction of the system, see Remark 2.5, and therefore such an observable has a jump at the transition for $\alpha > 1$.

If instead $\alpha \in (0, 1)$ formula (2.9) still holds, but the asymptotic behavior of the left-hand side is gotten by Riemann sum approximation:

$$\begin{aligned} \sum_n (1 - \exp(-b(h)n)) K(n) &\stackrel{h \searrow 0}{\sim} b^\alpha C_K b \sum_n \frac{1 - \exp(-b(h)n)}{(bn)^{1+\alpha}} \\ &\sim b^\alpha C_K \int_0^\infty \frac{1 - \exp(-x)}{x^{1+\alpha}} dx = b^\alpha C_K \frac{\Gamma(1-\alpha)}{\alpha}, \end{aligned} \quad (2.10)$$

so that it suffices to invert the asymptotic relation $(b(h))^\alpha C_K (\Gamma(1-\alpha)/\alpha) \stackrel{h \searrow 0}{\sim} h$ and this of course gives that $b(h)$ is asymptotically proportional to $h^{1/\alpha}$.

The arguments that we have developed directly lead to the following statement (see [29, Ch. 2] for a more complete statement and for more details on the proof):

Theorem 2.2. *The critical behavior of the free energy is given by*

$$F(0, h_c(0) + \delta) \stackrel{\delta \searrow 0}{\sim} \begin{cases} c_1 \delta & \text{if } \alpha > 1, \\ c_2 \delta^{1/\alpha} & \text{if } \alpha < 1, \end{cases} \quad (2.11)$$

with

$$c_1 := \frac{1 - K(\infty)}{\sum_{n \in \mathbb{N}} n K(n)} \quad \text{and} \quad c_2 := \left(\frac{\alpha(1 - K(\infty))}{C_K \Gamma(1-\alpha)} \right)^{1/\alpha}. \quad (2.12)$$

Moreover in full generality, as $N \rightarrow \infty$, $\mathbf{P}_{N,h}^c$ (that denotes the measure $\mathbf{P}_{N,\omega}^c$ when $\beta = 0$) converges weakly in the product topology of $\mathbb{R}^{\mathbb{N}}$ to a limit measure \mathbf{P}_h . The limit process

is a $K_{F(0,h)}(\cdot)$ -renewal, namely:

$$\mathbf{P}_h(\tau_1 = \ell_1, \tau_2 = \ell_1 + \ell_2, \dots, \tau_k = \ell_1 + \dots + \ell_k) = \prod_{j=1}^k K_{F(0,h)}(\ell_j), \quad (2.13)$$

where

$$K_{F(0,h)}(n) = \begin{cases} \exp(h - nF(0,h))K(n), & \text{if } F(0,h) > 0, \\ \exp(h)K(n), & \text{if } F(0,h) = 0. \end{cases} \quad (2.14)$$

Therefore $F(0,h) > 0$ implies that the limit process is positive persistent (in fact, the inter-arrival distribution decays exponentially), while instead if $F(0,h) = 0$ the limit inter-arrival distribution has power law decay and, if $h < h_c(0)$, the $K_{F(0,h)}(\cdot)$ -renewal is terminating.

Remark 2.3. Once sharp results for the constrained case are obtained, one can deduce sharp results on the free case. This is just based on the elementary formula

$$Z_{N,h}^f = \sum_{n=0}^N Z_{n,h}^c \bar{K}(N-n), \quad (2.15)$$

that we have written in the case $\sum_{n \in \mathbb{N}} K(n) = 1$ and we have introduced

$$\bar{K}(N) := \sum_{n \in \mathbb{N}: n > N} K(n). \quad (2.16)$$

For example if $h > h_c(0)$ from (2.4) we have

$$Z_{N,h}^f = \exp(F(0,h)N) \sum_{n=0}^N \mathbf{P}_h(n \in \tau) \exp(-F(0,h)(N-n)) \bar{K}(N-n). \quad (2.17)$$

Since $\exp(-F(0,h)(N-n)) \bar{K}(N-n)$ is bounded below by $\sum_{j > N-m} \exp(-F(0,h)j) K(j)$ and since $\sum_{n=0}^N \mathbf{P}_h(n \in \tau) \bar{K}(N-n) = 1$ for any persistent $K(\cdot)$ -renewal we obtain that for every N

$$Z_{N,h}^f \geq \exp(-h) \exp(F(0,h)N). \quad (2.18)$$

A (rough) bound in the other direction is obtained by neglecting $\mathbf{P}_h(n \in \tau) \bar{K}(N-n)$ in the right-hand side of (2.17), so that

$$Z_{N,h}^f \leq \frac{1}{1 - F(0,h)} \exp(F(0,h)N), \quad (2.19)$$

which holds once again for every N . The sharp asymptotic result is

$$\begin{aligned} Z_{N,h}^f &\stackrel{N \rightarrow \infty}{\sim} \frac{\exp(F(0,h)N)}{\sum_n n K_{F(0,h)}(n)} \sum_{n=0}^N \exp(-F(0,h)n) \bar{K}(n) \\ &\sim \frac{(1 - \exp(-h)) \partial_h F(0,h)}{1 - \exp(-F(0,h))} \exp(F(0,h)N). \end{aligned} \quad (2.20)$$

This type of estimates directly leads to computing the limit behavior of $\mathbf{P}_{N,h}^f$, see [29, Ch. 2] for details.

Remark 2.4. A key concept in statistical mechanics (and a key concept here) is the notion of *correlation length*. For example a natural correlation length of the system for $h > h_c(0)$ is given by the rate of exponential decay, as $n \rightarrow \infty$, of $\mathbf{P}_h(n \in \tau)$ to its limit value $1/\mathbf{E}_h[\tau_1]$. One can show [30] in particular that if h is sufficiently close to $h_c(0)$ then $\mathbf{P}_h(n \in \tau) - 1/\mathbf{E}_h[\tau_1] > 0$ and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\mathbf{P}_h(n \in \tau) - \frac{1}{\mathbf{E}_h[\tau_1]} \right) = -F(0, h), \quad (2.21)$$

which says that the correlation length coincides with $1/F(0, h)$. Even if one takes a *finite volume viewpoint*, $1/F(0, h)$ appears as a natural correlation length, for example because in (2.4) one sees that is only when N is of the order of $1/F(0, h)$ that one starts observing the exponential growth of the partition function. One could push these arguments a bit further and see that if N is *much smaller* than $1/F(0, h)$ (of course this has a precise sense only when $h \searrow h_c(0)$) then $\mathbf{P}_{N,h}^c$ resembles \mathbf{P} , while for N *much larger* than $1/F(0, h)$ the measure $\mathbf{P}_{N,h}^c$ starts exhibiting localization. The fact that the inverse of the free energy is the correlation length still holds also in presence of disorder [35, 52, 53], even if a full understanding of this important issue is still elusive.

Remark 2.5. The density of contacts, or contact fraction, that is the limit as $N \rightarrow \infty$ of $N^{-1}\mathbf{E}_h[\sum_{n=1}^N \mathbf{1}_{n \in \tau}]$ coincides by Theorem 2.2 with $\lim_{n \rightarrow \infty} \mathbf{P}_h^c(n \in \tau) = 1/\mathbf{E}_h\tau_1$. Note moreover that $N^{-1}\mathbf{E}_h[\sum_{n=1}^N \mathbf{1}_{n \in \tau}] = N^{-1}\partial_h \log Z_{N,h}^c$, so that the contact fraction is equal to $\partial_h F(0, h)$ (except, possibly, at $h = h_c(0)$). Moreover, by simple conditioning arguments one easily sees for example that for $h \neq h_c(0)$

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \max_{n: M \leq n \leq N-M} \left| \mathbf{P}_{N,h}^c(n \in \tau) - \frac{1}{\mathbf{E}_h\tau_1} \right| = 0, \quad (2.22)$$

and, by arguing like in Remark 2.3, one directly sees that the same statement holds for the free case.

3. THE DISORDERED CASE

3.1. The quenched free energy. An elementary observation that turns out to be really crucial for us at several instances is that for every $M = 0, 1, \dots, N$

$$\log Z_{N,\omega}^c \geq \log Z_{M,\omega}^c + \log Z_{N-M,\theta^M\omega}^c. \quad (3.1)$$

It is simply proven by restricting the renewal trajectories, in the expression for $Z_{N,\omega}^c$, to the ones that contain the contact site M and by using the renewal property. By averaging over the disorder one sees that $\{\mathbb{E} \log Z_{N,\omega}^c\}_{N=0,1,\dots}$ is super-additive and this entails [43] the existence of the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^c =: F(\beta, h), \quad (3.2)$$

and the limit of this sequence coincides with its supremum:

$$F(\beta, h) = \sup_{N \in \mathbb{N}} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^c. \quad (3.3)$$

The super-additive property (3.1) can however be exploited further in order to obtain results about the limit of the non averaged sequence $\{\log Z_{N,\omega}^c\}_{N=0,1,\dots}$, by applying the tools available for super-additive ergodic sequences, and notably the celebrated Kingman's Theorem [43]. Alternatively one can stick to the super-additive character of $\{\mathbb{E} \log Z_{N,\omega}^c\}_{N=0,1,\dots}$

and establish a concentration property of the non averaged sequence, either by using concentration inequalities, e.g. [45, 51], or even by more elementary tools [29, Ch. 4]. In all cases the result that one obtains is the existence and the self-averaging character of the free energy of pinning systems:

Theorem 3.1. *The sequence $\left\{N^{-1} \log Z_{N,\omega}^c\right\}_{N=0,1,\dots}$ converges to $F(\beta, h)$ both $\mathbb{P}(d\omega)$ -almost surely and in the L^1 sense.*

Remark 3.2. It is not difficult [29, Ch. 4] to show that for every $K(\cdot)$, β and h there exists $c > 0$ such that

$$Z_{N,\omega}^c \leq Z_{N,\omega}^f \leq c N Z_{N,\omega}^c, \quad (3.4)$$

uniformly in ω . We can therefore restate Theorem 3.1 replacing the superscript c with f .

Another elementary central fact is that

$$Z_{N,\omega}^c \geq \mathbf{E} \left[\exp \left(\sum_{n=1}^N (\beta \omega_n + h) \mathbf{1}_{n \in \tau} \right); \tau \cap [1, N] = \{N\} \right] = \exp(\beta \omega_N + h) K(N), \quad (3.5)$$

and therefore

$$F(\beta, h) \geq 0. \quad (3.6)$$

We now partition the parameter space of the system into:

$$\mathcal{L} := \{(\beta, h) : F(\beta, h) > 0\} \quad \text{and} \quad \mathcal{D} := \{(\beta, h) : F(\beta, h) = 0\}. \quad (3.7)$$

\mathcal{L} and \mathcal{D} stand respectively for *Localized* and *Delocalized* regime, a nomenclature that calls for further explanations (see § 3.2 just below), but for the moment we just point out that one of our main aim is to characterize these regions as precisely as possible. And a substantial help is given by the fact that the function $(\beta, h) \mapsto F(\beta, h)$ is convex, as limit of convex functions, and it is monotonic non decreasing in both variables (monotonicity in h is immediate, in β it is instead a consequence of convexity and of the fact that $\partial_h \mathbb{E} \log Z_{N,\omega}^c = 0$ for $\beta = 0$). Since by (3.6) we see that \mathcal{D} coincides with $\{(\beta, h) : F(\beta, h) = 0\}$ so that \mathcal{D} is a convex set.

One can go beyond: Jensen inequality (*annealing*) yields

$$\mathbb{E} \log Z_{N,\omega}^c \leq \log \mathbb{E} Z_{N,\omega}^c = \log \mathbf{E} \left[\exp \left((h + \beta^2/2) \sum_{n=1}^N \mathbf{1}_{n \in \tau} \right); N \in \tau \right], \quad (3.8)$$

so that

$$F(\beta, h) \leq F(0, h + \beta^2/2), \quad (3.9)$$

and if we recall that $F(\beta, h) \geq F(0, h)$ we directly get

$$h_c^{ann}(\beta) := h_c(0) - \frac{\beta^2}{2} \leq h_c(\beta) \leq h_c(0). \quad (3.10)$$

As a matter of fact the upper bound can be made strict, that is $h_c(\beta) < h_c(0)$ as soon as $\beta > 0$, in full generality (the generality here refers to the choice of $K(\cdot)$, [6]) and in the framework that we consider here one can show also that, given $K(\cdot)$, for every $\beta_0 > 0$ one can find an explicit constant $c \in (0, 1/2)$ such that $h_c(\beta) \leq h_c(0) - c\beta^2$ for $\beta \in (0, \beta_0]$ [29, Ch. 5]. Instead, showing that $h_c(\beta) > h_c^{ann}(\beta)$ is a more delicate issue (and it is not true in general!). These bounds are summed up in Figure 4 and they imply that, since \mathcal{D} is a convex set, then $h_c(\cdot)$ is concave and, since it is bounded, it is continuous.

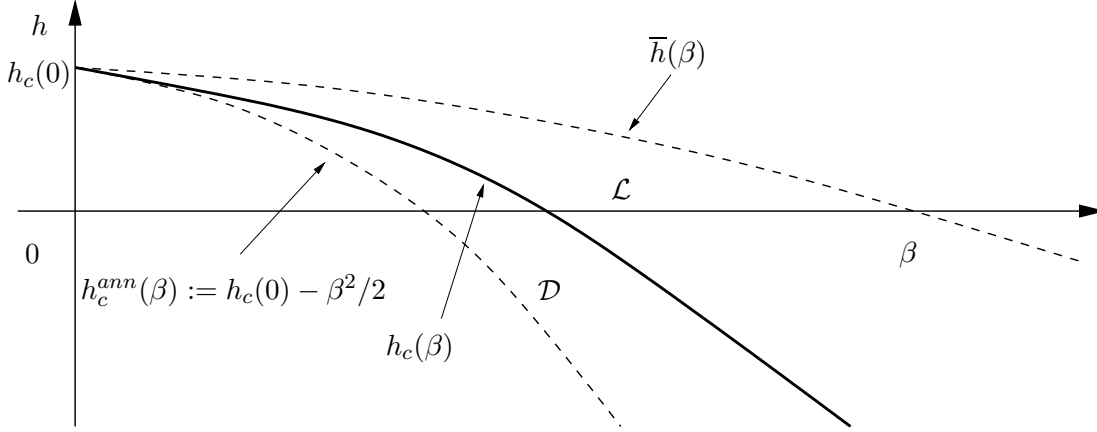


FIGURE 4. The critical curve $\beta \mapsto h_c(\beta)$ that separates \mathcal{D} and \mathcal{L} is concave decreasing. This follows from the fact that \mathcal{D} is a convex set and from the explicit bounds we have on the critical curve. The upper bound $\bar{h}(\beta)$ is less explicit than the lower bound, but we stress that $\bar{h}(\beta) < h_c(0)$ for every $\beta > 0$ and this shows that disorder may induce localization and it never suppresses it. The lower bound comes from the standard annealing procedure. Note that the annealed partition function $\mathbb{E}Z_{N,\omega}^a$, $a = \mathbf{c}, \mathbf{f}$, is just the homogeneous partition function with pinning potential $h + \beta^2/2$.

Let us sum up the outcome of the arguments we have just outlined:

Proposition 3.3. *If we set $h_c(\beta) = \inf\{h : F(\beta, h) > 0\}$ then $h_c(\beta) = \sup\{h : F(\beta, h) = 0\}$ and*

$$\mathcal{L} = \{(\beta, h) : h > h_c(\beta)\} \quad \text{and} \quad \mathcal{D} = \{(\beta, h) : h \leq h_c(\beta)\} \quad (3.11)$$

Moreover the function $\beta \mapsto h_c(\beta)$ is concave, decreasing and (3.10) holds for every β .

Remark 3.4. From (3.3) we actually extract the important observation that localization can be observed in finite volume, in the sense that $(\beta, h) \in \mathcal{L}$ if and only if there exists N such that $\mathbb{E} \log Z_{N,\omega}^c > 0$.

3.2. On path behavior. Characterizing localization and delocalization simply by looking at whether the free energy is positive or zero may look, from a mathematical standpoint, rather cheap. This is not the case as one can first see by observing that

$$\partial_h \frac{1}{N} \log Z_{N,\omega}^c = \mathbf{E}_{N,\omega}^c \left[\sum_{n=1}^N \mathbf{1}_{n \in \tau} \right], \quad (3.12)$$

which, by exploiting the convexity of the free energy and Theorem 3.1, tells us that $\mathbb{P}(d\omega)$ -a.s.

$$\partial_h F(\beta, h) = \lim_{N \rightarrow \infty} \mathbf{E}_{N,\omega}^c \left[\frac{1}{N} \sum_{n=1}^N \mathbf{1}_{n \in \tau} \right], \quad (3.13)$$

when $\partial_h F(\beta, h)$ exists. By convexity, such a derivative exists except possibly at a countable set of points and in any case (3.13) can be extended to a (standard) suitable statement also if the derivative does not exist, in terms of right and left derivatives [29] (as a matter of fact, in [35] it is shown that $F(\beta, \cdot)$ is C^∞ except possibly at $h_c(\beta)$). In the end (3.13) is telling us that the *contact fraction*, i.e. the right-hand side in (3.13), of our system is

zero if $h < h_c(\beta)$ (that is, in the interior of \mathcal{D}) and it is positive if $h > h_c(\beta)$ (that is, in the whole of \mathcal{L}). By itself this fully justifies our definition of (de)localization.

However (3.13) is still a poor result and plenty of questions could be asked about the limit of the sequence $\{\mathbf{P}_{N,\omega}^a\}_{N=0,1,\dots}$, $a = \mathbf{c}$ or $a = \mathbf{f}$, starting with the existence of such a limit. And of course the question is: how close one can get to the very sharp description of the limit measure available for homogeneous systems?

Work has been done in this direction, but we will not concentrate on this aspect. We just point out that

- (1) The localized phase is, to a certain extent, rather well understood. In the sense that if $(\beta, h) \in \mathcal{L}$ then one can show that the weak limit as N tends to infinity of the sequence of probability measures $\{\mathbf{P}_{N,\omega}\}_N$ exists $\mathbb{P}(\mathrm{d}\omega)$ -a.s. and the limit process is a point process with a positive density of points [35]. One can show also other estimates going toward the completely clear picture that emerges from the homogeneous case. Intriguing differences however do arise, naturally connected to the existence of exceptional deviations in the sequence of charges. Moreover a number of open questions still stand (see e.g. [36]).
- (2) Progress has been made only recently on the delocalized phase, at least away of criticality [32] (see [52] for some estimates at criticality). Essentially one now knows that in the delocalized non critical regime the number of contacts for a system of size N is $O(\log N)$ and such a result has been achieved by a subtle argument combining concentration bounds and super-additivity properties of $\log Z_{N,\omega}^{\mathbf{c}}$. Such a result still leaves open intriguing questions in the direction, for example, of the precise results proven in [16, 39] in the homogeneous or weakly inhomogeneous context, see for example in the bibliographic complements at the end of [29, Ch. 8].

3.3. The role of disorder. The main questions we want to address are:

- (1) How does the disorder affect the phase diagram? Namely can we determine $h_c(\beta)$, for $\beta > 0$, beyond the bounds in Figure 4?
- (2) What can one say about the critical behavior of the free energy? This amounts to estimating how $F(\beta, h)$ vanishes as $h \searrow h_c(\beta)$.

It is particularly interesting to raise such questions because we know $h_c(0)$ and we know the sharp asymptotic behavior of $F(0, h)$ for h close to $h_c(\beta)$ (see Theorem 2.2), so that in our framework inquiring about the role of the disorder makes perfect sense. At this point it is important to underline that such questions do find partial (non rigorous) answers in the physical literature: the rest of this subsection is devoted to explaining what one expects on the basis of formal expansions, following some renormalization group ideas. We must say that the arguments that follow are adaptation to the pinning model context of an argument developed by A. B. Harris [37] in the context the Ising model with random bond defects. Harris' argument is based on the idea that the behavior of a system near criticality should become rather independent of fine details, so in particular one can replace the system by a coarse grained one without changing substantially the properties. What one actually tries to do is defining a renormalization transformation, like decimation or block summation, that, once applied repeatedly at criticality, transform the system into a limit model. Harris' work aims at determining whether introducing the disorder modifies the fixed point of the renormalization transformation: if the renormalization transformation suppresses the disorder and the limit point is like in the homogeneous case, then one says that disorder is irrelevant. If instead disorder is enhanced one says that disorder is relevant

and most probably the renormalization transformation flow leads to a fixed point which is different from the one obtained in the homogeneous case. It should be noted on one hand that at the border between relevance and irrelevance the renormalization transformation, to first order, neither decreases nor increases the disorder: this is the so called marginal case. On the other hand, Harris argument is just a small disorder expansion and as such it does not apply to the whole range of the parameters and, above all, it does not characterize the limit fixed point when disorder is relevant.

Harris' ideas have been first applied in the pinning model context by G. Forgacs, J. M. Luck, Th. M. Nieuwenhuizen and H. Orland [25] and then by B. Derrida, V. Hakim and J. Vannimenus [20] with predictions that differ somewhat in a sense that we are going to explain just below.

Let us start with an expansion that is freely inspired by [25]. Without loss of generality we assume $h_c(0) = 0$ (recall (1.14)). Moreover the argument does not feel the boundary condition: we work it out in the free case. In what follows $\delta := h + \beta^2/2 \geq 0$: this change of variable is a natural one because in particular

$$\mathbb{E}[Z_{N,\omega}^f] = \mathbf{E} \left[\exp \left(\delta \sum_{n=1}^N \mathbf{1}_{n \in \tau} \right) \right] = Z_{N,\delta}^f. \quad (3.14)$$

We set $\zeta_n = \exp(\beta\omega_n - \beta^2/2) - 1$ and let us note that

$$\begin{aligned} \mathbb{E} \log \frac{Z_{N,\omega}^f}{\mathbb{E} Z_{N,\omega}^f} &= \mathbb{E} \log \mathbf{E}_{N,\delta}^f \left[\exp \left(\sum_{n=1}^N (\beta\omega_n - \beta^2/2) \mathbf{1}_{n \in \tau} \right) \right] \\ &= \mathbb{E} \log \mathbf{E}_{N,\delta}^f \left[\prod_{n=1}^N (1 + \zeta_n \mathbf{1}_{n \in \tau}) \right] \\ &= \mathbb{E} \log \left(1 + \sum_n \zeta_n \mathbf{P}_{N,\delta}^f(n \in \tau) + \sum_{n_1 < n_2} \zeta_{n_1} \zeta_{n_2} \mathbf{P}_{N,\delta}^f(\{n_1, n_2\} \subset \tau) + \dots \right). \end{aligned} \quad (3.15)$$

Let us now expand the logarithm and let us use the fact that the ζ random variables are centered and IID with variance equal to $\exp(\beta^2) - 1$ to see that

$$\mathbb{E} \log \frac{Z_{N,\omega}^f}{\mathbb{E} Z_{N,\omega}^f} = -\frac{1}{2} (\exp(\beta^2) - 1) \sum_{n=1}^N \mathbf{P}_{N,\delta}^f(n \in \tau)^2 + \dots \quad (3.16)$$

By Remark 2.5, for $\delta > 0$ and as long as n and $N - n$ are large, $\mathbf{P}_{N,\delta}^f(n \in \tau)$ is close to $\partial_\delta F(0, \delta)$ so that from (3.15) we extract

$$F(\beta, h_c^{ann}(\beta) + \delta) = F(\beta, \delta - \beta^2/2) = F(0, \delta) - \frac{1}{2} (\exp(\beta^2) - 1) (\partial_\delta F(0, \delta))^2 + \dots \quad (3.17)$$

Of course this expansion is only formal and in order to make it rigorous one has to control the rest. Let us note on the way that one can in principle try to compute all the terms in this expansion, but the issue of controlling the rest is still there and convergence issues may very well require β to be small (note that we are expanding using as small parameter the variance of ζ , but aiming at capturing the critical behavior, hence h is small too). All the same, (3.17) is compatible with $h_c(\beta) = h_c(0)$ if $F(0, \delta)$ vanishes much slower than $(\partial_\delta F(0, \delta))^2$ as $\delta \searrow 0$ (β possibly small, but fixed). But by Remark 3.13 (or directly by taking the h derivative in (2.2)) we see that $\partial_\delta F(0, \delta) = 1/\mathbf{E}_\delta \tau_1$ and by direct computation

(similar to (2.10)) one sees that $\partial_\delta F(0, \delta) \stackrel{\delta \searrow 0}{\sim} (c_2/\alpha) \delta^{-1+1/\alpha}$ for $\alpha \in (0, 1)$ (c_2 is given in Theorem 2.2, but the precise value does not play a role here), while the contact fraction is bounded away from zero when $\alpha > 1$ even approaching criticality. So (3.17) is compatible with $h_c(\beta) = h_c(0)$ if

$$\delta^{1/\alpha} \stackrel{\delta \searrow 0}{\gg} \delta^{2(-1+1/\alpha)} \iff \alpha < \frac{1}{2}. \quad (3.18)$$

This argument therefore suggests that disorder is irrelevant for $\alpha < 1/2$.

If $\alpha > 1/2$ the expansion we have performed looks hopeless, but we may argue that this is just due to the fact that $h_c(\beta) > h_c^{ann}(\beta)$ and we are expanding around the *wrong point*. Of course we do know that $F(\beta, h_c(\beta)) = 0$ and therefore (3.17) suggests that for β small the shift of the quenched critical point $\delta_c(\beta) := h_c(\beta) - h_c^{ann}(\beta)$ is found by equating the two terms in the rightmost side of (3.17) and this procedure suggests $\delta_c(\beta) \approx \beta^{2\alpha/(2\alpha-1)}$.

A second approach is instead inspired by [20]. If we aim at analyzing whether the annealed system is close to the quenched system one could sit at the annealed critical point ($h = h_c(0) - \beta^2/2 = -\beta^2/2$, i.e. $\delta = 0$) and study the variance of $Z_{N,\omega}^f$ (once again, the argument would go through also with constrained boundary condition). Divergence of the variance, as $N \rightarrow \infty$, would be a sign that quenched and annealed systems aren't close. Since at $\delta = 0$ we have $\mathbb{E} Z_{N,\omega}^f = 1$ and

$$\text{var}_{\mathbb{P}}(Z_{N,\omega}^f) = \mathbb{E} \left[(Z_{N,\omega}^f)^2 - 1 \right] = \mathbb{E} \mathbf{E}^{\otimes 2} \left[\exp \left(\sum_n (\beta \omega_n - \beta^2/2) (\mathbf{1}_{n \in \tau} + \mathbf{1}_{n \in \tau'}) \right) - 1 \right], \quad (3.19)$$

with τ and τ' independent copies of the same renewal process. Integrating out the ω variables we obtain

$$\text{var}_{\mathbb{P}}(Z_{N,\omega}^f) = \mathbf{E}^{\otimes 2} \left[\exp \left(\beta^2 \sum_{n=1}^N \mathbf{1}_{n \in \tau \cap \tau'} \right) - 1 \right]. \quad (3.20)$$

This expression can be evaluated in a sharp way because the random set $\tau \cap \tau'$ is still a renewal process and therefore the variance that we are evaluating is the partition function of a homogeneous pinning model (minus one). And the first relevant question is whether $\tau \cap \tau'$ is a terminating or a persistent renewal. The inter-arrival law of $\tau \cap \tau'$ can be expressed in terms of the inter-arrival law of τ only in an implicit way, but the renewal function of $\tau \cap \tau'$ is explicit in terms of the renewal function of τ :

$$\mathbf{P}^{\otimes 2}(n \in \tau \cap \tau') = \mathbf{P}(n \in \tau)^2, \quad (3.21)$$

and $\tau \cap \tau'$ is terminating (respectively, persistent) if $\sum_n \mathbf{P}(n \in \tau)^2 < \infty$ (respectively, $\sum_n \mathbf{P}(n \in \tau)^2 = \infty$) and, by Proposition 1.2, we see that

$$\gamma_2 := \sum_{n=1}^{\infty} \mathbf{P}(n \in \tau)^2 < \infty \iff \sum_n \frac{1}{n^{2(1-\alpha)}} < \infty \iff \alpha < \frac{1}{2}. \quad (3.22)$$

By the general solution of the homogeneous model, cf. Section 2, we see that if $\tau \cap \tau'$ is persistent, then for every $\beta > 0$ the variance of $Z_{N,\omega}^f$ grows exponentially, while if $\tau \cap \tau'$ is terminating then $X := |\tau \cap \tau'| - 1$ is a geometric random variable (this is just a consequence of the renewal property) of expectation γ_2 , that is $\mathbf{P}(X = n) = (\gamma_2/(1 + \gamma_2))^n (1/(1 + \gamma_2))$, $n = 0, 1, \dots$. Therefore, as long as $\beta < \beta_0 := \sqrt{\log((1 + \gamma_2)/\gamma_2)}$, with $p_2 := (1/(1 + \gamma_2))$

we have

$$\lim_{N \rightarrow \infty} \text{var}_{\mathbb{P}} (Z_{N,\omega}^f) = \frac{p_2}{1 - (1 - p_2) \exp(\beta^2)} - 1 = \gamma_2 \beta^2 + \dots \quad (3.23)$$

where the expansion is of β small. Therefore if $\tau \cap \tau'$ is terminating (note that τ and τ' are persistent since we are assuming $h_c(0) = 0$) the variance of $Z_{N,\omega}^f$, at the critical annealed point, stays bounded and it is small if β is small. To complement (3.23) note that

$$\sup_N \text{var}_{\mathbb{P}} (Z_{N,\omega}^f) \leq \frac{p_2}{1 - (1 - p_2) \exp(\beta^2)} - 1 \stackrel{\beta \leq \beta_0/2}{\leq} \tilde{c} \beta^2, \quad (3.24)$$

for some $\tilde{c} > 0$.

Let us sum up the outcome of the arguments we have just outlined:

- (1) Both approaches suggest that disorder is irrelevant if $\alpha < 1/2$ (and, as a consequence, relevant if $\alpha > 1/2$, with the case $\alpha = 1/2$ as marginal one), as one can read from (3.18) and (3.22). Moreover the arguments do suggest that the annealed system is very close to the quenched one, in particular $h_c(\beta) = h_c(0)$, at least for β not too large. This observation may be considered as the Harris criterion prediction for pinning models.
- (2) There is a difference between (3.18) and (3.22) in the case $\alpha = 1/2$ that we cannot appreciate since we are assuming (1.9). In the more general framework of Remark 1.4 one sees that the fact that $L(\cdot)$ diverges at infinity does not imply that $\tau \cap \tau'$ is terminating, while it is sufficient to conclude that $F(0, \delta)$ is much larger than $(\partial_\delta F(0, \delta))^2$ for δ small. As a matter of fact, we are dealing with the marginal case in the renormalization group sense. This is a very subtle issue, still unresolved even on a purely heuristic level. We should stress that the steps that we have just presented here are just a part of the arguments in [25] and [20], in particular [25] aims at an expansion to all orders and [20] contains a subtle attempt to study the renormalization group flow for δ close to 0. Both [25] and [20] consider only the case of $L(\cdot)$ asymptotically constant and, for this case, their predictions differ.

3.4. Relevance and irrelevance of the disorder: the results. The heuristic picture outlined in the previous subsection has now been made rigorous. A summary of these rigorous results is given in the next three theorems. We recall that $h_c^{ann}(\beta) = h_c(0) - \beta^2/2$ and that the annealed free energy is $F(0, h + \beta^2/2)$, so that the annealed critical behavior is obtained by looking at $F(0, h_c^{ann}(\beta) - (\beta^2/2) + \delta) = F(0, h_c(0) + \delta)$ as $\delta \searrow 0$.

Theorem 3.5. *Choose $\alpha \in (0, 1/2)$ and $K(\cdot)$ satisfying (1.8) and (1.9). Then there exists $\beta_0 > 0$ such that $h_c(\beta) = h_c^{ann}(\beta)$ for $\beta \leq \beta_0$. Moreover, for the same values of β the critical behavior of the quenched free energy coincides with the critical behavior of the annealed free energy:*

$$\log F(\beta, h_c(\beta) + \delta) \stackrel{\delta \searrow 0}{\sim} \log F(0, h_c(0) + \delta). \quad (3.25)$$

Theorem 3.5 has been first proven in [4] by using a modified second moment method that we are going to outline in Section 4. It has been then proven also in [54], by interpolation techniques. Both works contain more detailed results than just Theorem 3.5, in particular (3.25) has been established by showing that the stronger statement (4.1) holds. As a matter of fact in [36, Th. 2.3] it has been proven that

$$\lim_{\beta \searrow 0} \limsup_{\delta \searrow 0} \left| \frac{F(0, \delta) - F(\beta, h_c^{ann}(\beta) + \delta)}{(\beta^2/2)(\partial_\delta F(0, \delta))^2} - 1 \right| = 0, \quad (3.26)$$

written for $h_c(0) = 0$ for sake of compactness. Note that (3.26) is in agreement with (3.17) and in fact the first step in justifying that expansion.

Theorem 3.6. *Choose $K(\cdot)$ satisfying (1.8) and (1.9). If $\alpha > 1/2$ we have $h_c(\beta) > h_c^{ann}(\beta)$. Moreover*

- (1) *if $\alpha \in (1/2, 1)$ we have that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that*

$$h_c(\beta) - h_c^{ann}(\beta) \geq c_\varepsilon \beta^{\frac{2\alpha}{2\alpha-1} + \varepsilon}, \quad (3.27)$$

for every $\beta \leq 1$;

- (2) *for every $K(\cdot)$ such if $\alpha > 1$ there exists $c > 0$ such that*

$$h_c(\beta) - h_c^{ann}(\beta) \geq c\beta^2, \quad (3.28)$$

for every $\beta \leq 1$.

The results in Theorem 3.6 are *almost* sharp, because in [4, 54] it is proven that for every $K(\cdot)$ such that $\alpha \in (1/2, 1)$ there exists $C > 0$ such that

$$h_c(\beta) - h_c^{ann}(\beta) \leq C\beta^{\frac{2\alpha}{2\alpha-1}}, \quad (3.29)$$

for every $\beta \leq 1$. On the other hand the bound in Theorem 3.6(2) is already optimal (in the same sense) in view of the bounds summed up in the caption of Figure 4. The result (3.27) has now been improved to match precisely (3.29), i.e. it has been shown in [7] that in (3.27) one can take $\varepsilon = 0$ and c_0 is still positive.

Theorem 3.6 has been proven in [19] and we give an outline of the proof in Section 5. The method is based on estimating fractional moments of the free energy, while (3.29) is derived by adapting the techniques yielding Theorem 3.5 (and a sketch of the proof is in Section 4). In [19] the case $\alpha = 1$ is not considered, but in [10] it is shown that the fractional moment method can be generalized to establish in particular that $h_c(\beta) > h_c^{ann}(\beta)$ also for $\alpha = 1$.

For what concerns the critical behavior we have the following:

Theorem 3.7. *For every $K(\cdot)$ we have*

$$(0 \leq F(\beta, h) \Rightarrow) F(\beta, h) - F(\beta, h_c(\beta)) \leq \frac{1+\alpha}{\beta^2} (h - h_c(\beta))^2, \quad (3.30)$$

for every h (of course the result is non trivial only for $h > h_c(\beta)$).

The result in Theorem 3.7 has been established in [34] for rather general charge distribution. The proof that we give in Section 6 uses rather heavily the Gaussian character of the charges and it is close to the argument sketched in [33].

Let us point out that Theorem 3.7, coupled with Theorem 2.2, shows that the critical behavior of quenched and annealed systems differ as soon as $\alpha > 1/2$, in full agreement with the Harris criterion:

$$\liminf_{\delta \searrow 0} \frac{\log (F(\beta, h_c(\beta) + \delta) - F(\beta, h_c(\beta)))}{\log \delta} \stackrel{\alpha > 1/2}{>} \frac{\log (F(0, h_c(0) + \delta) - F(0, h_c(0)))}{\log \delta}, \quad (3.31)$$

since the left-hand side is bounded below by 2, by Theorem 3.7, and the right-hand side is equal to $\max(1, 1/\alpha)$, by Theorem 2.2.

Remark 3.8. Theorem 3.7 therefore shows that the disorder, for $\alpha > 1/2$, has a smoothing effect on the transition. The Harris criterion in principle is just suggesting that there is no reason to believe that the critical behavior is the same. There is a general belief that disorder smooths the transitions: this is definitely the case for a number of statistical mechanics models to which a celebrated result of M. Aizenman and J. Wehr applies [3] (see also [38]). It should be however remarked that the Aizenman-Wehr smoothing mechanism does not yield smoothing for the pinning model and that the argument leading to Theorem 3.7 is very different from the argument in [3] (for more on this issue see the caption of Figure 6).

Remark 3.9. The amount of smoothing proven by Theorem 3.7 was not fully expected. In fact in [48] it is claimed that for $\alpha > 1$ the transition is still of first order, in disagreement for example with [17, 18]. Needless to say that it would be very interesting to understand what is really the value of the exponent for $\alpha > 1/2$ and how it depends on α . In [5] it is shown that pinning models based on exponentially decaying inter-arrival laws may not exhibit smoothing.

Remark 3.10. The results we have presented do not consider the case $\alpha = 1/2$. The results in this case are incomplete and not conclusive under hypothesis (1.9). In [4, 54] it is shown that under (1.9) one has $h_c(\beta) - h_c^{ann}(\beta) \leq c \exp(-1/(c\beta^2))$ for some $c > 0$ and $\beta \leq 1$. This bound matches the prediction in [20]. It is not known whether $h_c(\beta) - h_c^{ann}(\beta) > 0$, leaving open the possibility for the prediction in [25], i.e. $h_c(\beta) = h_c^{ann}(\beta)$ for β small, to be the right one. One has to point out however that the approach in [25] is a development in powers of β that cannot capture contributions beyond all orders. This issue remains open and debated also at a heuristic level. About the critical behavior, the smoothing result in Theorem 3.7 is once again not conclusive (but it does imply for example that disorder smooths the transition as soon as $\lim_{n \rightarrow \infty} L(n) = 0$, if $L(\cdot)$ is chosen as in Remark 1.4). So, under assumption (1.9) (which is the one that arises in the basic example, cf. (1.4)), the issue of whether $\alpha = 1/2$ is *marginally relevant* or *marginally irrelevant* is open.

4. FREE ENERGY LOWER BOUNDS AND IRRELEVANT DISORDER ESTIMATES

This section is mostly devoted to giving the main ideas of the proof of Theorem 3.5, but in § 4.2 we will also explain why such arguments yield also (3.29).

4.1. The case of $\alpha < 1/2$: the irrelevant disorder regime. As we have already pointed out, the annealed bound already yields, in full generality, that $F(\beta, h) \leq F(0, h + \beta^2/2)$, and hence $h_c(\beta) \geq h_c(0) - \beta^2/2 = h_c^{ann}(\beta)$. In order to pin, for $\alpha < 1/2$, that $h_c(\beta) \leq h_c^{ann}(\beta)$ we need to prove a lower bound on the free energy showing that $F(\beta, h) > 0$ whenever $F(0, h + \beta^2/2) > 0$. We are actually aiming at capturing also the critical behavior of the free energy; in fact we are aiming at showing that for every $\varepsilon > 0$ there exists $\beta_\varepsilon > 0$ such that for every $\beta \in (0, \beta_\varepsilon)$ we have

$$\liminf_{h \searrow 0} \frac{F(\beta, h)}{F(0, h + \beta^2/2)} \geq 1 - \varepsilon, \quad (4.1)$$

which yields $h_c(\beta) \leq h_c^{ann}(\beta)$ and it is a result (sensibly) stronger than (3.25). Note that by what we have seen on the expansion of the free energy (3.17), or by the rigorous result (3.26), we cannot aim at proving that the quenched free energy coincides with the annealed one. This actually casts some doubts about the applicability of second moment methods.

As a matter of fact if we choose $h > -\beta^2/2$ ($h_c(0) = 0$), as usual with $\delta = h + \beta^2/2$ we can write

$$\frac{\text{var}_{\mathbb{P}}(Z_{N,\omega}^f)}{(\mathbb{E}Z_{N,\omega}^f)^2} = \mathbf{E}_{\delta}^{\otimes 2} \left[\exp \left(\beta^2 \sum_{n=1}^N \mathbf{1}_{n \in \tau \cap \tau'} \right) - 1 \right]. \quad (4.2)$$

Note the analogy with (3.20) formula, in which $\delta = 0$: this time, since the underlying measure is \mathbf{P}_h (see the beginning of Section 2 or Theorem 2.2), $\tau \cap \tau'$ is a positive persistent renewal and the expression in (4.2) is growing exponentially as $N \rightarrow \infty$ for every $\beta > 0$ (unlike for the $\delta = 0$ case, in which the exponential growth sets up only for β larger than a positive constant β_0). As we have pointed out, this was to be expected: can one still extract from (4.2) some interesting information? The answer is positive, as shown by K. Alexander in [4].

The crucial point is not to take the limit in N , but rather exploit (4.2) up to the scale of the correlation length of the annealed system, which is just a homogeneous system with pinning potential δ (see Remark 2.4). The idea is to establish that the quenched partition function is close to the annealed one with large probability up to the correlation length scale. Note that, as pointed out in Remark 2.4, on such a scale the annealed partition function starts exhibiting exponential growth and one feels the size of the free energy (we will actually need to choose the size of the system, call it N_0 , to be a large, but finite, multiple of the correlation length, the relative error ε in (4.1) leaves the room to make such an estimate). Once such an estimate on a system of length N_0 is achieved, it is a matter of chopping the polymer into N/N_0 portions: some work at the boundary of these regions is needed and for that we refer to [4], while we focus on explaining why the second moment method works up to the correlation length scale.

Let us therefore go back to (4.2) and let us set $N_0 := q/F(0, \delta)$ (and assume that it is an integer number). With such a choice, by exploiting the estimates outlined in Remark 2.3, it is not difficult to see that for every $K(\cdot)$ and $q > 0$ there exists $c_K(q) > 0$ such that

$$\mathbb{E}Z_{N_0,\omega}^f = \mathbf{E} \exp \left(\delta \sum_{n=1}^N \mathbf{1}_{n \in \tau} \right) \leq c_K(q). \quad (4.3)$$

This is because, as long as q is finite and δ tends to zero, the system is in the *critical window* (as a matter of fact, in [50] the limit of $\mathbb{E}Z_{N_0,\omega}^f$ as δ tends to zero, q kept fixed, is computed and the notion of critical window is further elaborated). The constant $c_K(q)$ of course diverges as $q \nearrow \infty$. On the other hand $\mathbb{E}Z_{N,\omega}^f \geq 1$ so that in order to estimate the quantity in (4.2) it suffices to estimate

$$\mathbf{E}^{\otimes 2} \left[\exp \left(\delta \sum_{n=1}^N (\mathbf{1}_{n \in \tau} + \mathbf{1}_{n \in \tau'}) \right) \left(\exp \left(\beta^2 \sum_{n=1}^N \mathbf{1}_{n \in \tau \cap \tau'} \right) - 1 \right) \right]. \quad (4.4)$$

We now use the Cauchy-Schwarz inequality and the fact that $(\exp(x) - 1)^2 \leq \exp(2x) - 1$ for $x \geq 0$ to bound the expression in (4.4) (and therefore the expression in (4.2)) by

$$\mathbf{E} \left[\exp \left(2\delta \sum_{n=1}^N \mathbf{1}_{n \in \tau} \right) \right] \mathbf{E} \left[\left(\exp \left(2\beta^2 \sum_{n=1}^N \mathbf{1}_{n \in \tau \cap \tau'} \right) - 1 \right) \right]^{1/2} =: T_1 \cdot T_2. \quad (4.5)$$

But, by (4.3), T_1 is bounded by a constant (which depends on q). On the other hand, T_2 has been already estimated in (3.20)-(3.24), and it is $O(\beta)$, thanks to the fact that the

renewal $\tau \cap \tau'$ is terminating (since $\alpha < 1/2$). Therefore, by Chebychev inequality, for every $\epsilon > 0$

$$\mathbb{P}(Z_{N_0, \omega}^f \geq (1 - \epsilon)\mathbb{E}Z_{N_0, \omega}^f) \leq \frac{C_K(q)}{\epsilon^2} \sqrt{\tilde{c}} \beta, \quad (4.6)$$

where $C_K(q)$ is a constant depending on $K(\cdot)$ and q (it is just the constant $c_K(q)$ of (4.3) when δ is replaced by 2δ) and \tilde{c} is taken from (3.24). Since $\mathbb{E}Z_{N_0, \omega}^f$ is bounded below by $\exp(-\delta) \exp(F(0, \delta)N_0) = \exp(-\delta + q)$ (see (2.18)) we see that on the scale of correlation length the quenched partition function grows (almost) like the annealed one with large probability if β is small enough.

4.2. Lower bounds on the free energy beyond the irrelevant disorder regime.

The technique for lower bounds on the free energy that we have outlined, as well as the technique in [54, 56], lead to upper bounds on $h_c(\beta)$ also in the case $\alpha \in [1/2, 1)$, see (3.29) and Remark 3.10. But a look at Theorem 3.6(2) suffices to see that the case $\alpha > 1$ is somewhat different, because it is no longer true that $h_c(\beta) - h_c^{ann}(\beta) = o(\beta^2)$. So, in this case, the easy bound $h_c(\beta) \leq h_c(0)$ (which is just a consequence of convexity) is *optimal* in the sense that $h_c(\beta) - h_c^{ann}(\beta) \leq \beta^2/2 = O(\beta^2)$.

Let us therefore explain why the second moment method yields also (3.29) when $\alpha \in (1/2, 1)$. For this we go back to (4.2) and (4.4) (we are still placing ourselves on the scale of the correlation length). The term T_1 is still bounded by a constant, that of course depends on $K(\cdot)$ and q , just as in the $\alpha < 1/2$ case. The term T_2 this time grows exponentially in N , because this time $\tau \cap \tau'$ is persistent, and we have to worry about the size of N also for this term. But let us quickly estimate the growth rate of T_2 and for which values of N we can expect this term to be small for β small. A necessary condition, that with some careful work one can show also to be sufficient, is that the expectation of the term in the exponent in the exponent is small, namely that (cf. Proposition 1.2)

$$\beta^2 \sum_{n=1}^N \mathbf{P}(n \in \tau)^2 \stackrel{N \rightarrow \infty}{\sim} c_\alpha \beta^2 N^{2\alpha-1}, \quad (4.7)$$

has to be chosen small. However we still keep $N = N_0 = q/F(0, \delta)$, that is N of the order of $\delta^{-1/\alpha}$ (by Theorem 2.2), times a constant which is large if q is large. Plugging such a value of N in (4.7) we see that we are asking $\beta^2 \delta^{(1-2\alpha)/(2\alpha)}$ to be small. Therefore in this regime we expect the second moment method to work, leading to localization and also to the fact that the quenched free energy is fairly close to the annealed one, if

$$\delta \geq c \beta^{2\alpha/(2\alpha-1)}, \quad (4.8)$$

with c a small (fixed) constant. But this what is claimed in (3.29).

5. RELEVANT DISORDER ESTIMATES: CRITICAL POINT SHIFT

Annealing is the standard procedure to get upper bounds on disordered partition functions. One can go beyond by partial annealing procedures, like the *constrained* annealing procedure [44], and this does give some results, see e.g. [4], but for pinning models constrained annealing, in the infinite volume limit, yields nothing beyond the annealed bound if we are concerned with identifying the critical point [15]. There is therefore the need for a different idea.

5.1. Fractional moment estimates. A tool that allows to go beyond the annealed bound $h_c(\beta) \geq h_c^{ann}(\beta)$ (we set $h_c(0) = 0$ also in this section) turns out to be estimating $A_N := \mathbb{E}[(Z_{N,\omega}^c)^\gamma]$ for $\gamma \in (0, 1)$ by means of the basic inequality

$$\left(\sum_j a_j \right)^\gamma \leq \sum_j a_j^\gamma, \quad (5.1)$$

that holds whenever $a_j \geq 0$ for every j . This has been pointed out by F. L. Toninelli in [55]. Inequality (5.1) has been exploited also in other contexts, notably in [22, 14], to get upper bounds on the partition function of the directed polymer in random environment, and in [2] to establish localization of eigenfunctions for random operators, in particular in the Anderson localization context.

For pinning models one applies (5.1) to the renewal identity

$$Z_{N,\omega}^c = \sum_{n=0}^{N-1} Z_{n,\omega}^c K(N-n) \xi_N, \quad \text{with } \xi_N := \exp(\beta \omega_N + h), \quad (5.2)$$

and, by taking the expectation, one gets to

$$A_N \leq \mathbb{E}[\xi_1^\gamma] \sum_{n=0}^{N-1} A_n K(N-n)^\gamma = \sum_{n=1}^N A_{N-n} Q(n), \quad (5.3)$$

where $Q_n := \mathbb{E}[\xi_1^\gamma] K(n)^\gamma$. Now the point is that (5.3) implies

$$A_N \leq \left(\sum_{n=1}^{\infty} Q(n) \right) \max_{n=0,1,\dots,N-1} A_n, \quad (5.4)$$

so that, if $\sum_n Q(n) \leq 1$ we have $A_N \leq A_0 = 1$ for every N . Summing everything up

$$\mathbb{E}[\xi_1^\gamma] \sum_{n=1}^{\infty} K(n)^\gamma \leq 1 \implies \sup_N A_N \leq 1. \quad (5.5)$$

And of course if A_N has sub-exponential growth the free energy is zero since

$$\frac{1}{N} \mathbb{E} \log Z_{N,\omega}^c = \frac{1}{\alpha N} \mathbb{E} \log (Z_{N,\omega}^c)^\alpha \leq \frac{1}{\alpha N} \log A_N. \quad (5.6)$$

Remark 5.1. The discrete convolution inequality (5.5) can actually be exploited more. Observe in fact that the solution to the renewal equation

$$u_0 = 1 \quad \text{and} \quad u_N = \sum_{n=1}^N Q(n) u_{N-n} \quad \text{for } N = 1, 2, \dots, \quad (5.7)$$

dominates A_N . But if $Q(\cdot)$ is a probability distribution (possibly adding $Q(\infty)$), then u is the renewal function of the $Q(\cdot)$ -renewal and if $\sum_n Q(n) < 1$ then $u_N \sim cQ(N)$ for $N \rightarrow \infty$ (c is an explicit constant, see Theorem 1.2). Therefore

$$\mathbb{E}[\xi_1^\gamma] \sum_{n=1}^{\infty} K(n)^\gamma < 1 \implies \text{there exists } C > 0 \text{ such that } A_N \leq CK(N)^\gamma. \quad (5.8)$$

We are now left with verifying for which values of β and h we can find γ such that $\mathbb{E}[\xi_1^\gamma] \sum_{n=1}^\infty K(n)^\gamma \leq 1$. Let us consider the case $\sum_n K(n) = 1$: in this case $\sum_n K(n)^\gamma > 1$, so the question is for which values of β and h the pre-factor $\mathbb{E}\xi_1^\gamma$ is sufficiently small. This is a straightforward computation ($\delta = h + \beta^2/2$):

$$\mathbb{E}\xi_1^\gamma = \exp\left(-\frac{\beta^2}{2}\gamma(1-\gamma) + \delta\gamma\right), \quad (5.9)$$

which is small for β sufficiently large, for every fixed value of δ . This result is therefore saying that (it may be helpful to keep in mind Figure 4):

- (1) $h_c(\beta) > h_c^{ann}(\beta)$ if β is sufficiently large;
- (2) the gap between $h_c(\beta)$ and $h_c^{ann}(\beta)$ becomes arbitrarily large as β tends to infinity: in fact it is of the order of β^2 .

Note that this approach yields very explicit bounds, but it does not give results for small values of β .

5.2. Iterated fractional moment estimates. To go beyond the estimate we have just presented, in [19] another renewal identity has been exploited, namely: for every fixed k and every $N \geq k$

$$Z_{N,\omega}^c = \sum_{n=k}^N Z_{N-n,\omega}^c \sum_{j=0}^{k-1} K(n-j) \xi_{N-j} Z_{j,\theta^{N-j}\omega}^c. \quad (5.10)$$

This is simply obtained by decomposing the constrained partition function according to the value $N-n$ of the last point of τ before or at $N-k$ ($0 \leq N-n \leq N-k$ in the sum), and to the value $N-j$ of the first point of τ to the right of $N-k$ (so that $N-k < N-j \leq N$). Of course $Z_{j,\theta^{N-j}\omega}^c$ has the same law as $Z_{j,\omega}^c$ and the three random variables $Z_{N-n,\omega}^c$, ξ_{N-j} and $Z_{j,\theta^{N-j}\omega}^c$ are independent, if $n \geq k$ and $j < k$.

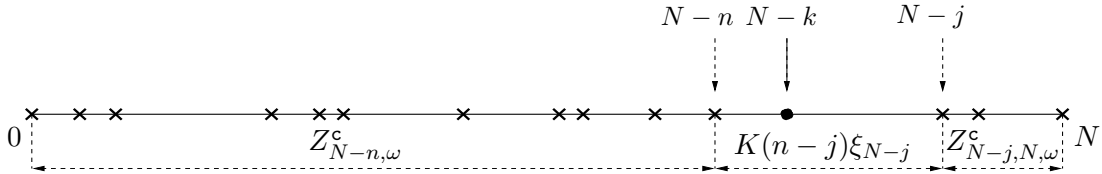


FIGURE 5. The renewal identity (5.10) is obtained by fixing a value of k and summing over the values of the last contact before $N-k$ (the large dot in the figure) and the first contact after $N-k$. The two contacts are respectively $N-n$ and $N-j$ (crosses are contacts in the figure).

Let $0 < \gamma < 1$, set once again $A_N := \mathbb{E}[(Z_{N,\omega}^c)^\gamma]$ and use (5.1) and (5.10) to get for $N \geq k$

$$A_N \leq \mathbb{E}[\xi_1^\gamma] \sum_{n=k}^N A_{N-n} \sum_{j=0}^{k-1} K(n-j)^\gamma A_j. \quad (5.11)$$

This is still a renewal type inequality since it can be rewritten as

$$A_N \leq \sum_{n=1}^N A_{N-n} Q_k(n), \quad (5.12)$$

with $Q_k(n) := \mathbb{E}[\xi_1^\gamma] \sum_{j=0}^{k-1} K(n-j)^\gamma A_j$ if $n \geq k$ and $Q_k(n) := 0$ for $n < k$. In particular if for given β and h one can find $k \in \mathbb{N}$ and $\gamma \in (0, 1)$ such that

$$\rho := \sum_n Q_k(n) = \mathbb{E}[\xi_1^\gamma] \sum_{n=k}^{\infty} \sum_{j=0}^{k-1} K(n-j)^\gamma A_j \leq 1, \quad (5.13)$$

then one directly extracts from (5.12) that

$$A_N \leq \rho \max\{A_0, \dots, A_{N-k}\}, \quad (5.14)$$

for $N \geq k$, which implies that $A_N \leq \max\{A_0, \dots, A_{k-1}\}$ and hence $F(\beta, h) = 0$.

Remark 5.2. Like in Remark 5.1 one can be sharper by exploiting the renewal structure in (5.12). The difference with Remark 5.1 is that in this case $N \geq k$. In order to put (5.12) into a more customary *renewal form* we set $\tilde{A}_N := A_N \mathbf{1}_{N \geq k}$, so that

$$\tilde{A}_N \leq \sum_{n=1}^{N-(k-1)} \tilde{A}_{N-n} Q_k(n) + P_k(N), \quad \text{with } P_k(N) = \sum_{n=0}^{k-1} A(n) Q_k(N-n), \quad (5.15)$$

and therefore there exists $c > 0$ (depending on k , $K(\cdot)$ and γ , besides of course β and h) such that $P_k(N) \leq c Q_k(N)$. Let us now consider the standard renewal equation for the $Q_k(n)$ -renewal: $u_0 = 1$ (but of course one can choose an arbitrary $u_0 > 0$) and

$$u_N = \sum_{n=1}^N u_{N-n} Q_k(n) \stackrel{N \geq k}{=} \sum_{n=1}^{N-(k-1)} u_{N-n} Q_k(n) + u_0 Q_k(N), \quad (5.16)$$

where the first equality holds for $N = 1, 2, \dots$ and for the second one we have used that, since $Q_k(n) = 0$ up to $n = k-1$, we have $u_1 = u_2 = \dots = u_{k-1} = 0$. Once again if $\sum_n Q_k(n) < 1$, that is if $\rho < 1$, we are dealing with a renewal equation of a terminating process and therefore u_N behaves asymptotically like (a constant times) $Q_k(N)$. Comparing (5.15) and (5.16) one obtains that there exists a constant $C = C(K(\cdot), k, \gamma, h, \beta)$ such that

$$A_N \leq C K(N)^\gamma. \quad (5.17)$$

5.3. Finite size estimates by shifting. What the iterated fractional moment has done for us is reducing the problem of estimating the free energy from above to a finite volume estimate. Notice in fact that estimating ρ , cf. (5.13), amounts to estimating only (a fractional moment of) $Z_{j,\omega}^c$, for $j < k$ (note the parallel with Remark 3.4!). This type of estimates demands a new ingredient, which is more easily explained when $\alpha > 1$. A preliminary observation is that ρ is bounded above by $\varepsilon^{-1} \sum_{j=0}^{k-1} (A_j / (k-j)^{(1+\alpha)\gamma-1})$, with ε a constant that depends on $K(\cdot)$ and γ so that $\rho \leq 1$ is implied if for a given γ

$$\sum_{j=0}^{k-1} \frac{A_j}{(k-j)^{(1+\alpha)\gamma-1}} \leq \varepsilon. \quad (5.18)$$

Note that, unlike the case treated in § 5.1, here the pre-factor $\mathbb{E}\xi_1^\gamma$, and therefore β and h , has only a marginal role: as long as β and h are chosen in a compact, which is what we are doing since we are focusing on the critical region of the annealed model at small or moderate values of β , ε can be chosen independent of the value of β and h . We shall see that the expression in (5.18) can be made small for example by choosing k large.

Let us start by observing that we know of course (Jensen inequality) that for $h = h_c^{ann}(\beta) + \delta$

$$A_j \leq (\mathbb{E} Z_{j,\omega}^c)^\gamma = \left(\mathbb{E} \left[\exp \left(\delta \sum_{n=1}^j \mathbf{1}_{n \in \tau} \right); j \in \tau \right] \right)^\gamma = \exp(\gamma F(0, \delta) j) \mathbf{P}_\delta(j \in \tau)^\gamma. \quad (5.19)$$

We are of course interested in $\delta > 0$: by the Renewal Theorem $\mathbf{P}_\delta(j \in \tau)$ is bounded below by a positive constant (even if δ were zero!), so this term cannot be of much help and we simply bound it above by one. On the other hand the exponentially growing term stays bounded for j up to the correlation length of the annealed system (cf. Remark 2.4): we therefore choose $k := 1/F(0, \delta)$ (again, assume that it is in \mathbb{N}). At this point we observe that we can choose $\gamma \in (0, 1)$ such that

$$(1 + \alpha)\gamma > 2, \quad (5.20)$$

then the expression in (5.18) is bounded for k large, that is δ small. This is not yet what we want, but a more attentive analysis shows that one has

$$\sum_{j=0}^{k-1-R} \frac{A_j}{(k-j)^{(1+\alpha)\gamma-1}} \leq \exp(\gamma) \sum_{j>R} j^{-(1+\alpha)\gamma+1} \leq \varepsilon/2, \quad (5.21)$$

for any $k \geq R$ and R chosen sufficiently large (depending only on γ , α and ε). This has been achieved by using (5.19). We have therefore to show that

$$\sum_{j=k-R}^{k-1} \frac{A_j}{(k-j)^{(1+\alpha)\gamma-1}} \leq \varepsilon/2. \quad (5.22)$$

For this we set

$$\hat{A}_k := \limsup_{\delta \searrow 0} \max_{j=k-R, \dots, k-1} A_j. \quad (5.23)$$

If we are able to show that

$$\hat{A}_k \sum_{i=1}^R i^{-((1+\alpha)\gamma-1)} \leq \varepsilon/3, \quad (5.24)$$

then (5.18) would be established (for δ small and $k = 1/F(0, \delta)$). Of course in (5.24) one can replace R with ∞ obtaining thus a more stringent condition (but, in the end, equivalent, since we are not tracking the constants). For a proof of (5.24) one has to go beyond (5.19) and in doing so the size of β turns out to play a role.

In order to go beyond (5.19) the new idea is a tilting procedure (first proposed in [31]), that, given the Gaussian context, reduces to a shift. The idea is based on the following consequence of Hölder inequality

$$A_j = \mathbb{E}' \left[(Z_{j,\omega}^c)^\gamma \frac{d\mathbb{P}}{d\mathbb{P}'}(\omega) \right] \leq \mathbb{E}' [Z_{j,\omega}^c]^\gamma \mathbb{E}' \left[\left(\frac{d\mathbb{P}}{d\mathbb{P}'}(\omega) \right)^{1/(1-\gamma)} \right]^{1-\gamma}, \quad (5.25)$$

where \mathbb{P}' is a probability with respect to which \mathbb{P} is absolutely continuous. In order to make the choice of \mathbb{P}' let us fix $\delta = a\beta^2$, a a constant that we are going to choose along the way: \mathbb{P}' is the law of the sequence

$$\omega_1 - \sqrt{a\beta^2}, \omega_2 - \sqrt{a\beta^2}, \dots, \omega_k - \sqrt{a\beta^2}, \omega_{k+1}, \omega_{k+2}, \dots \quad (5.26)$$

which is a sequence of independent (non identically distributed) variables. One then readily computes

$$\mathbb{E}' \left[\left(\frac{d\mathbb{P}}{d\mathbb{P}'}(\omega) \right)^{1/(1-\gamma)} \right]^{1-\gamma} = \exp \left(\frac{\gamma}{1-\gamma} a \beta^2 k \right), \quad (5.27)$$

but $a\beta^2 k = \delta/F(0, \delta)$ and the ratio $\delta/F(0, \delta)$ tends to a positive constant as $\delta \searrow 0$ since $\alpha > 1$, cf. Theorem 2.2. Let us now turn our attention to $\mathbb{E}' Z_{j,\omega}^c$ which, for $j \leq k$, coincides with $\mathbb{E}' Z_{j,\omega-\sqrt{a\beta^2}}^c$. But this is just the partition function of a homogeneous model with negative pinning potential if we choose a small, namely (for conciseness we look only at the case $j = k$)

$$\begin{aligned} \mathbb{E}' Z_{k,\omega}^c &= \mathbf{E} \left[\exp \left(-\beta^2 (\sqrt{a} - a) \sum_{n=1}^k \mathbf{1}_{n \in \tau} \right); k \in \tau \right] \\ &= \mathbf{E} \left[\exp \left(- \left(\frac{1}{\sqrt{a}} - 1 \right) \left(\frac{\delta}{F(0, \delta)} \right) \frac{1}{k} \sum_{n=1}^k \mathbf{1}_{n \in \tau} \right); k \in \tau \right]. \end{aligned} \quad (5.28)$$

But $\lim_{k \rightarrow \infty} (1/k) \sum_{n=1}^k \mathbf{1}_{n \in \tau} = 1/\mathbf{E}[\tau_1]$ \mathbf{P} -a.s. and this readily implies that $\mathbb{E}' Z_{k,\omega}^c$ is bounded by a constant that can be chosen arbitrarily small, provided one chooses a sufficiently small (so $a^{-1/2} - 1$ is large). The argument easily extends to j between $k - R$ and k so that (5.24) is proven since \hat{A}_k can be chosen arbitrarily small for a sufficiently small and every $\delta \leq \delta_0$ (for some $\delta_0 > 0$). This concludes the argument for the case $\alpha > 1$, that is Theorem 3.6(2).

The case $\alpha \in (1/2, 1)$ (Theorem 3.6(1)) is conceptually not very different: the main difference lies in the fact that it is no longer sufficient to show that A_j is small for j close to k , one has actually to extract some decay in j . But the fact that A_j does decay with j , at least if $j \leq 1/F(0, \delta)$, is already rather evident from (5.19) from the fact that the term $\mathbf{P}_\delta(j \in \tau)$ is, at least till $j < k = 1/F(0, \delta)$, close to $\mathbf{P}(j \in \tau)$ which behaves for j large as $j^{-(1-\alpha)}$ (times a constant, cf. Theorem 1.2). The argument is however somewhat technical and we refer to [19] for details.

6. RELEVANT DISORDER ESTIMATES: THE CRITICAL EXPONENT

The argument leading to Theorem 3.7 is based on the *rare stretch* strategy sketched in Figure 6. It is based on a one-step coarse graining of the environment on the scale $\ell \in \mathbb{N}$, $1 \ll \ell \ll N$. Actually one should think of ℓ as very large but finite. We assume $N/\ell \in \mathbb{N}$ and we look at the sequence of IID random variables defined as

$$Y_j := \mathbf{1}_{E_j}, \quad \text{with } E_j = \left\{ \omega : \log Z_{\ell, \theta(j-1)\ell, \omega}^c \geq aF(\beta, h + \delta) \right\}, \quad (6.1)$$

where $\delta > 0$, $a \in (0, 1)$ (eventually $a \nearrow 1$) and $j = 1, 2, \dots$, but of course only the j 's up to N/ℓ are relevant to us. Note that the Y variables are Bernoulli random variables of parameter $p(\ell) := \mathbb{P}(E_1)$ and, since $a < 1$, $p(\ell)$ is small when ℓ is large, by the very definition of the free energy and its self-averaging property (cf. Theorem 3.1). One can actually show rather easily that

$$\liminf_{\ell \rightarrow \infty} \frac{1}{\ell} \log p(\ell) \geq -\frac{\delta^2}{2\beta^2}. \quad (6.2)$$

We give a proof of this inequality below, but the intuitive reason is that the probability of observing $\sum_{i=1}^{\ell} \omega_i \approx \ell\delta/\beta$ behaves like $\exp(-\ell\delta^2/2\beta^2)$ for ℓ large (this is the standard Cramer Large Deviation result). When such a Large Deviation event occurs, the environment in the ℓ -block will look like the original ω variables translated of δ/β , that is $\beta\omega_i + h$ looks like $\beta\omega_i + h + \delta$. And in that block the logarithm of the partition function will hence be close to $\ell F(\beta, h + \delta)$. Shifting the mean is of course only one possible strategy to make the event E_1 typical and hence such an argument yields only a lower bound on $p(\ell)$.

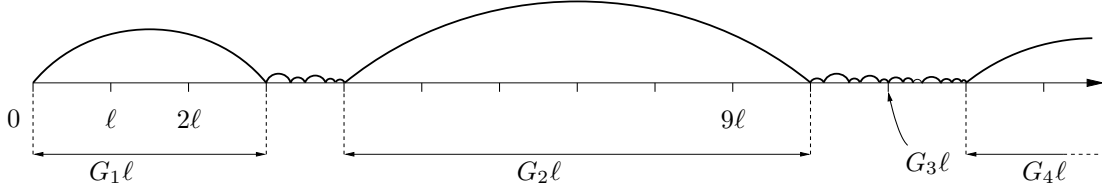


FIGURE 6. The rare stretch strategy is implemented by looking at blocks of ω variables of size ℓ . To block j is associated a Bernoulli random variable Y_j : such a random variable is a function of the ω variables in the block and it determines whether a system of the size of that block, with constrained boundary conditions and precisely with the ω variables of the block, has a sufficiently large (in fact, an atypically large) partition function. The precise definition of Y_j is in (6.1). In the figure $Y_4 = Y_{11} = Y_{12} = 1$, while the other Y variables are zero. The gaps between the success blocks are parametrized as $G_n\ell$, so that $\{G_n\}_n$ are IID Geometric random variables: $\mathbb{P}(G_1 = k) = (1-p)^k p$ ($k = 0, 1, 2, \dots$ and $p = p(\ell)$, given in the text). The lower bound is then achieved by restricting the partition function to the τ trajectories that visit only success blocks and visit the first and the last point of the block (also if two success blocks are contiguous: this is the case of the second and third success blocks in the figure). Such a strategy is profoundly different from that employed in [3], that is based on the effect of typical fluctuations (on the Central Limit Theorem scale) and competition with boundary effects. Our strategy is instead a Large Deviation strategy and, in a sense, it exploits the *flexibility* of the polymer to target rare regions (the boundary conditions play no role).

We now make a lower bound on $Z_{N,\omega}^f$ by considering only the τ -trajectories that visit all and only the ℓ blocks for which $Y_j(\omega) = 1$ (see Figure 6). The renewal property leads to a rather explicit lower bound, namely (with the notation of the figure)

$$\log Z_{N,\omega}^f \geq \sum_{\substack{j \leq N/\ell: \\ Y_j=1}} \log Z_{\ell, \theta(j-1)\ell_\omega}^c + \sum_{n=1}^{\mathcal{N}_Y(\omega)} \log K(G_n\ell) + O(\log N), \quad (6.3)$$

where the $O(\log N)$ term comes from the last excursion. Let us now divide by N and take the limit, keeping into account that, by definition of the Y variables, we have a lower bound on the partition functions $\log Z_{\ell, \theta(j-1)\ell_\omega}^c$ that appear in the right-hand side. We therefore obtain

$$\begin{aligned} F(\beta, h) &\geq a\ell F(\beta, h + \delta) \lim_{N \rightarrow \infty} \frac{\mathcal{N}_Y(\omega)}{N} + \limsup_{N \rightarrow \infty} \left(\frac{\mathcal{N}_Y(\omega)}{N} \frac{1}{\mathcal{N}_Y(\omega)} \sum_{n=1}^{\mathcal{N}_Y(\omega)} \log K(G_n\ell) \right) \\ &= ap(\ell)F(\beta, h + \delta) + \frac{p(\ell)}{\ell} \limsup_{N \rightarrow \infty} \frac{1}{\mathcal{N}_Y(\omega)} \sum_{n=1}^{\mathcal{N}_Y(\omega)} \log K(G_n\ell), \end{aligned} \quad (6.4)$$

where in the last step we have used the strong law of large numbers (the limits are in the $\mathbb{P}(d\omega)$ -a.s. sense) to estimate the leading behavior of the number of successes in an array of N/ℓ Bernoulli variables of parameter $p(\ell)$. Moreover, to be precise, (6.4) holds if we set $K(0) = 1$ (which we do only here). The (superior) limit that is left in the expression is also easily evaluated by using the strong law of large numbers after having observed that, since $\log K(x) \stackrel{x \rightarrow \infty}{\sim} -(1 + \alpha) \log x$, when $G_n = 1, 2, \dots$ we have $\log K(G_n \ell) \geq -a^{-1}(1 + \alpha)(\log G_n + \log \ell)$ for ℓ sufficiently large (uniformly in the value of G_n : note that $1/a$ is once again just a number larger than 1). The net outcome is therefore

$$F(\beta, h) \geq ap(\ell)F(\beta, h + \delta) - \frac{p(\ell)}{\ell}a^{-1}(1 + \alpha)(\mathbb{E}[\log G_1; G_1 > 0] + \mathbb{P}(G_1 > 0) \log \ell), \quad (6.5)$$

Since G_1 is a geometric variable of parameter $p(\ell)$ we directly compute $\mathbb{E}[\log G_1; G_1 > 0] = (1 + o_\ell(1)) \log(1/p(\ell))$ which, by (6.2), is bounded above by $a^{-1}\ell\delta^2/(2\beta^2)$ (ℓ large: once again a^{-1} is just used as an arbitrary constant larger than one). Therefore

$$F(\beta, h) \geq p(\ell) \left[aF(\beta, h + \delta) - a^{-2} \frac{\delta^2(1 + \alpha)}{2\beta^2} + o_\ell(1) \right], \quad (6.6)$$

where the term $o_\ell(1)$ is $-c(\log \ell)/\ell$ ($c > 0$) and this bound holds, given $a \in (0, 1)$, for every ℓ larger than some ℓ_0 .

Now we set $h = h_c(\beta)$ in (6.6), so that the left-hand side is zero and therefore

$$aF(\beta, h_c(\beta) + \delta) - a^{-2} \frac{\delta^2(1 + \alpha)}{2\beta^2} + o_\ell(1) \leq 0, \quad (6.7)$$

for every $\ell > \ell_0$, so that $F(\beta, h_c(\beta) + \delta) \leq a^{-3}(\delta^2(1 + \alpha)/(2\beta^2))$, and since $a \in (0, 1)$ is arbitrary we can let $a \nearrow 1$ and we are done.

For completeness we give a proof of (6.2). We call $\tilde{\mathbb{P}}_\ell$ the law of the sequence of random variables

$$\omega_1 + \delta/\beta, \omega_2 + \delta/\beta, \dots, \omega_\ell + \delta/\beta, \omega_{\ell+1}, \omega_{\ell+2}, \dots \quad (6.8)$$

Note that $\mathbb{P}_\ell(E_1)$ tends to one as ℓ becomes large, simply by definition of free energy and because the law of $\{\beta\omega_n + h\}_{n=1, \dots, \ell}$, when ω is distributed according to $\tilde{\mathbb{P}}_\ell$, coincides with the law of $\{\beta\omega_n + h + \delta\}_{n=1, \dots, \ell}$, when ω is distributed according to \mathbb{P} . We compute the relative entropy

$$\mathcal{H}(\tilde{\mathbb{P}}_\ell | \mathbb{P}) := \tilde{\mathbb{E}}_\ell \left[\log \frac{d\tilde{\mathbb{P}}_\ell}{d\mathbb{P}}(\omega) \right] = \frac{\delta^2}{2\beta^2} \ell, \quad (6.9)$$

and by a standard entropy inequality (see e.g. [29, § A.2])

$$\log \left(\frac{\mathbb{P}(E_1)}{\tilde{\mathbb{P}}_\ell(E_1)} \right) \geq -\frac{1}{\tilde{\mathbb{P}}_\ell(E_1)} \left(\mathcal{H}(\tilde{\mathbb{P}}_\ell | \mathbb{P}) + \frac{1}{e} \right) = -\frac{1}{\tilde{\mathbb{P}}_\ell(E_1)} \left(\frac{\delta^2}{2\beta^2} \ell + \frac{1}{e} \right), \quad (6.10)$$

and this yields (6.2).

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